Friends, Strangers, Math, and Graphs

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STEAM Week by Superposition Fremont

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1. Some fun math

   - Friends and Strangers

2. The life of a mathematician

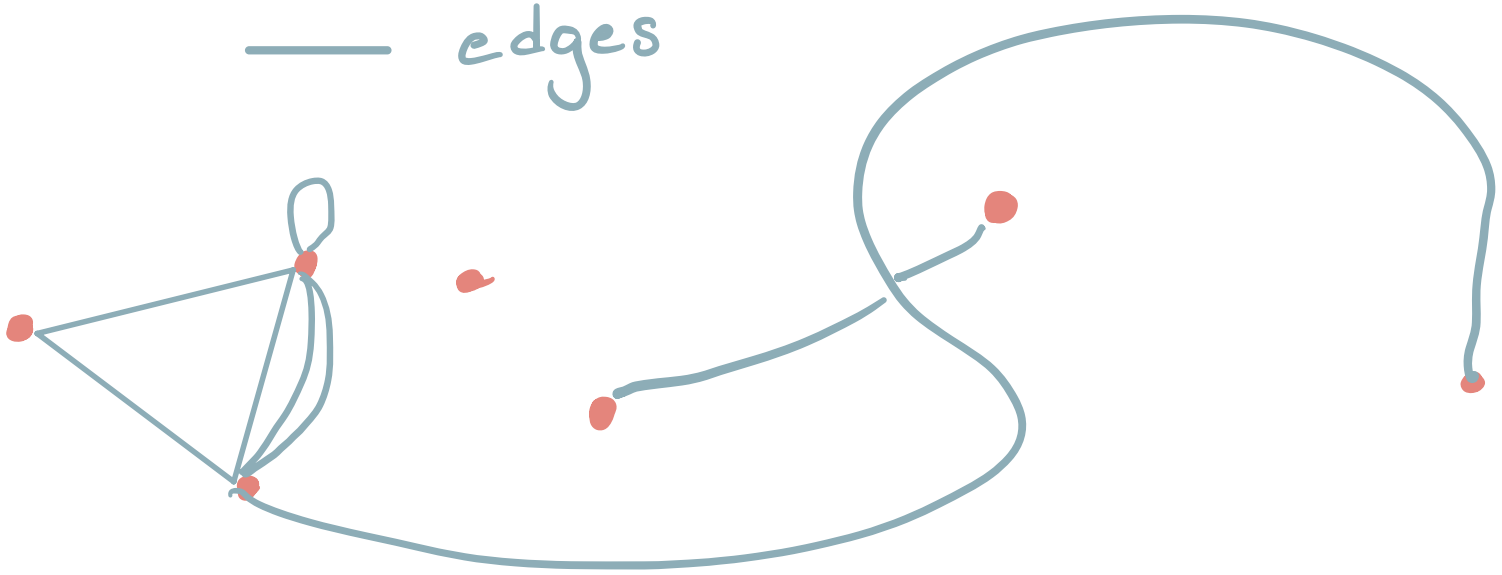
   - What does it mean to be a mathematician?
   - How could I become a mathematician?
   - Can I become a mathematician?

Graphs

Graphs are objects made up of 2 things:

- vertices
- edges

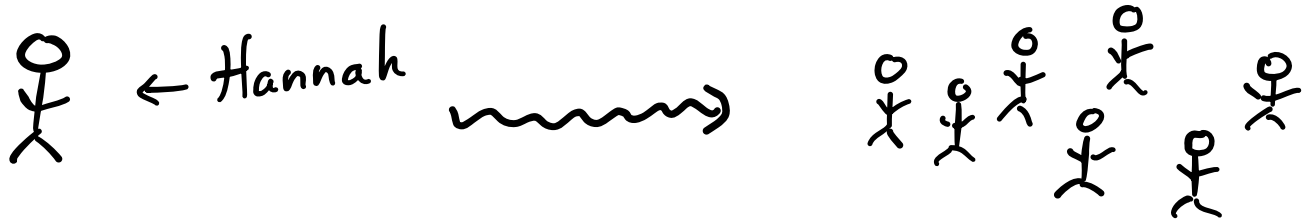
Ex.



A counting problem

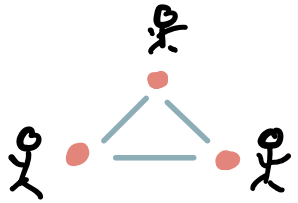
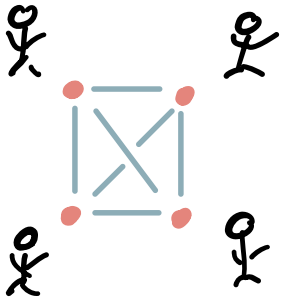
(Combinatorics)

Question:



~ Hannah goes to a party with 7 people ~

How many fist bumps are possible if noone fist bumps another person more than once?

People	Graph	# Fist bumps
2		1
3		3
4		6

Punchline. Using graphs, we can encode data! & we can count things!

Answer. If n = # people, then

$$\#\text{ possible fist bumps} = \frac{n(n-1)}{2}$$

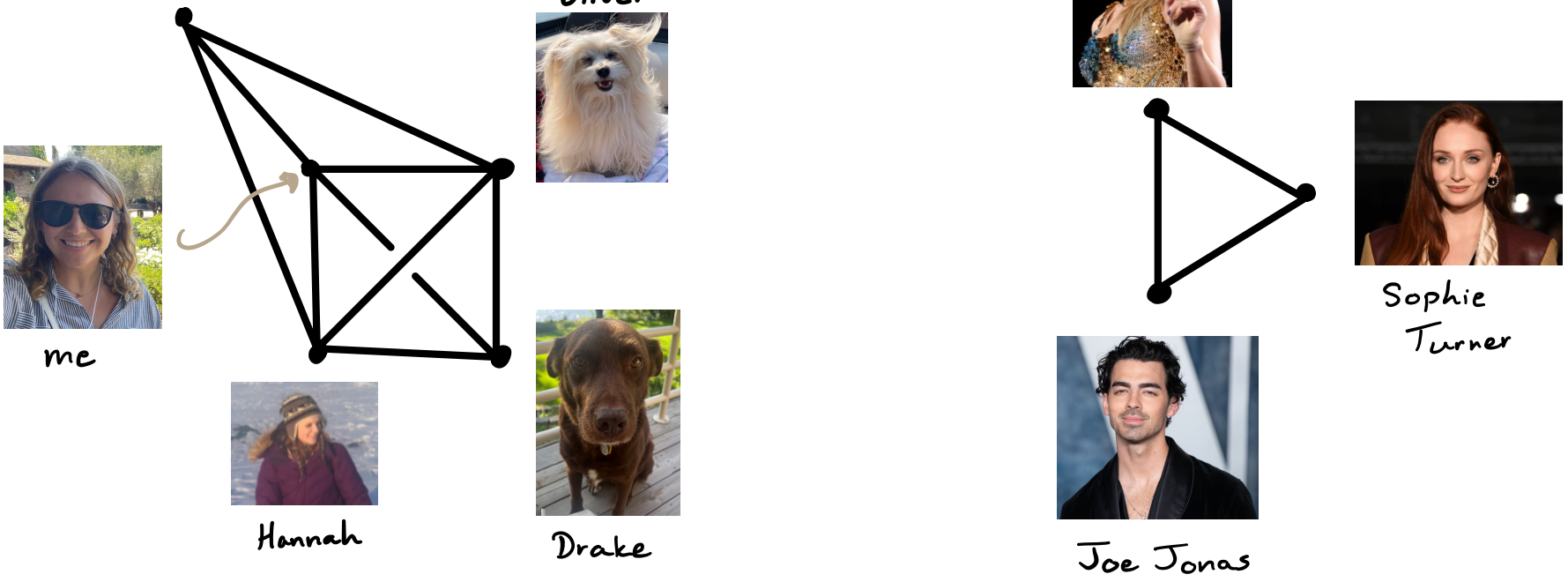
So if there are 8 people at Hannah's party, then

$$\frac{8(8-1)}{2} = 28.$$

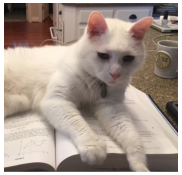
Ramsey theory

Friends and Strangers graphs

Colin Defant
Noah Kravitz

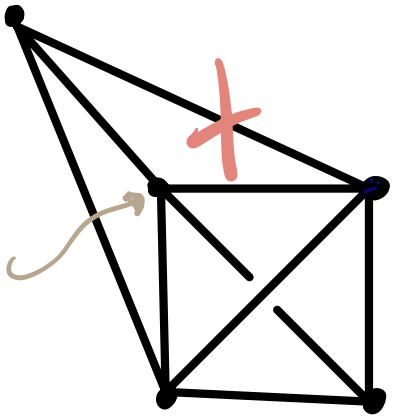


Slalom



Drake

me

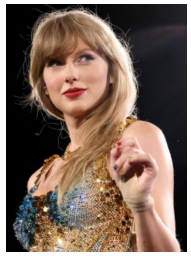


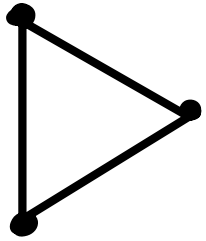
Hannah

Oliver

this is not a friendly swap

Taylor Swift





Joe Jonas

Sophie Turner

a friendly swap

Question.

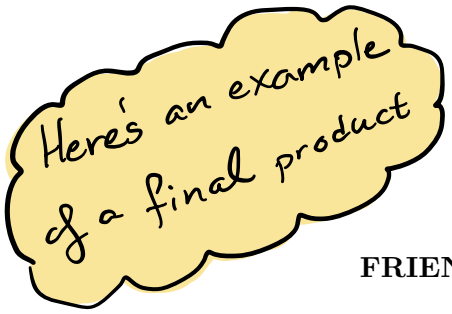
Open Problem!

When is it possible to get from one configuration (way people are standing on the graph) to another configuration?

Punchline.

Research can look like:

- Choosing an object
- Defining an action on that object
- Asking questions!



FRIENDS AND STRANGERS WALKING ON GRAPHS

COLIN DEFANT AND NOAH KRAVITZ

Abstract. Given graphs X and Y with vertex sets $V(X)$ and $V(Y)$ of the same cardinality, we define a graph $\mathsf{FS}(X,Y)$ whose vertex set consists of all bijections $\sigma\colon V(X)\to V(Y)$, where two bijections σ and σ' are adjacent if they agree everywhere except for two adjacent vertices $a,b\in V(X)$ such that $\sigma(a)$ and $\sigma(b)$ are adjacent in Y. This setup, which has a natural interpretation in terms of friends and strangers walking on graphs, provides a common generalization of Cayley graphs of symmetric groups generated by transpositions, the famous 15-puzzle, generalizations of the 15-puzzle as studied by Wilson, and work of Stanley related to flag h-vectors. We derive several general results about the graphs $\mathsf{FS}(X,Y)$ before focusing our attention on some specific choices of X. When X is a path graph, we show that the connected components of $\mathsf{FS}(X,Y)$ correspond to the acyclic orientations of the complement of Y. When X is a cycle, we obtain a full description of the connected components of $\mathsf{FS}(X,Y)$ in terms of toric acyclic orientations of the complement of Y. We then derive various necessary and/or sufficient conditions on the graphs X and Y that guarantee the connectedness of $\mathsf{FS}(X,Y)$. Finally, we raise several promising further questions.

1. Introduction

Let X be a simple graph with n vertices. Imagine that n different people, any two of whom are either friends or strangers, are standing so that one person is at each vertex of X. At each point in time, two friends standing at adjacent vertices of X may swap places by simultaneously crossing the edge that connects their respective vertices; two strangers may not swap places in this way. Our goal is to understand which configurations of people can be reached from other configurations when we allow the people to swap places repeatedly in this manner. The resolution of this problem certainly depends on the graph X, as well as on the information about which people are friends with each other; this further information can be encoded in a graph Y whose vertex set is the set of people and where edges indicate friendship.

To formalize and refine this problem, we define the *friends-and-strangers graph* $\mathsf{FS}(X,Y)$ whose vertex set is the set of bijections $\sigma\colon V(X)\to V(Y)$. Two bijections $\sigma,\sigma'\colon V(X)\to V(Y)$ are adjacent in $\mathsf{FS}(X,Y)$ if and only if we can find an edge $\{a,b\}$ in X such that:

- $\{\sigma(a),\sigma(b)\}$ is an edge in Y;
- $\sigma(a)=\sigma'(b)$ and $\sigma(b)=\sigma'(a)$;
- $\sigma(c)=\sigma'(c)$ for all $c\in V(X)\setminus\{a,b\}$.

When this is the case, we refer to the operation that transforms σ into σ' as an *(X,Y)-friendly swap across $\{a,b\}$*. Performing an (X,Y)-friendly swap corresponds to allowing two friends to swap places in the graph X. Notice that the isomorphism type of $\mathsf{FS}(X,Y)$ depends only on the isomorphism types of X and Y. Since we will usually be concerned only with the graph-theoretic structure of $\mathsf{FS}(X,Y)$ (such as the number and sizes of connected components), we will often specify the graphs X and Y only up to isomorphism.

It is sometimes convenient to assume that $V(X)$ and $V(Y)$ are both the set $[n]:=\{1,\dots,n\}$. In this case, the vertices of $\mathsf{FS}(X,Y)$ are the elements of the symmetric group $\mathfrak{S}_n$, which consists of all permutations of the numbers $1,\dots,n$. For $i,j\in[n]$, let $(i\,j)$ be the transposition in $\mathfrak{S}_n$ that swaps the numbers i and j. If $\sigma\in\mathfrak{S}_n$ is such that $\{i,j\}$ is an edge in X and $\{\sigma(i),\sigma(j)\}$ is an edge in Y, then we can perform an (X,Y)-friendly swap across $\{i,j\}$ to change σ into the permutation $\sigma\circ(i\,j)$. If we write the permutation σ in one-line notation as $\sigma=\sigma(1)\cdots\sigma(n)$, then

So you want to be a mathematician?

- What does it mean to be a mathematician?
- What do you do?
- How could I become a mathematician?
- Can I become a mathematician?
- How much school do I need?
- How do I get into grad school?

⋮

College

- Directed Reading Programs (DRP)
- Research Experience for Undergraduates:
  https://www.nsf.gov/crssprgm/reu/
- Computer science (CS) skills are useful, so take some CS courses if you can!
- Take courses that interest you because math has many applications.

Remember: Your mental health comes first!

Post-Baccalaureate Programs

- These are for people who want to go to grad school for math but feel like they need more background.
- These are great!

Graduate School

- 5-6 years
- Funded by fellowships or teaching (20k-50k)
- The first 2-3 years you take courses and some programs require that you pass 2-3 written exams on particular topics.
- At this point you have your masters and can leave if you are unhappy or if your goals have changed.
- The last years you find an advisor (a professor) who helps you write a disertation (a big research paper).

Post-doc

- 1-6 years
- If they're shorter you may do 2-3.
- Funded by grants or by teaching
- This is when you try to put together a research program and really figure out who you are as a mathematician.
- Give talks on your research

- If you only want to teach, you can skip this step and apply to be a professor at a liberal arts school.

Professor

- As long as you want once you get tenure.
- Teach courses
- If youre at a research institution, you work on your research and advise graduate students.
- Participate in comittees to help the math dept. run.
- Go to conferences and give talks on your research.
- ... and much more.

But remember:

- You are unique and will make your own unique path!

- It's always o.k. to decide the path you are on is not right for you and leave any program you are in at any time, even if the change seems weird to other people.

- You don't need to be in academia to enjoy math!

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