Group Actions on Stanley and Stanley-Reisner Rings



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# Overview

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**Definition:** A **simplicial complex** $\Delta$ on a vertex set $V = V(\Delta)$ is a collection of subsets $F \subseteq V$ (called faces) such that

1. if $v \in V$, then $\{v\} \in \Delta$,
2. and if $F \in \Delta$ and $G \subset F$, then $G \in \Delta$

- The **dimension** of $F \in \Delta$ is $|F| - 1$.
- $f(\Delta) = (f_{-1}, f_0, \ldots, f_d)$ where $f_i = \#\{\text{faces of dimension } i\}$ is called the **$f$-vector** of $\Delta$.

**Definition:** If $V = \{x_1, \ldots, x_n\}$ is the vertex set of a simplicial complex $\Delta$, then the **Stanley-Reisner ring** is the quotient ring $\Bbbk[\Delta] = \Bbbk[V]/I_\Delta$ where

$$I_\Delta = \langle x_{i_1} \cdots x_{i_r} : x_{i_j} \in V \text{ but } \{x_{i_1}, \ldots, x_{i_r}\} \notin \Delta \} \rangle.$$

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# A Parameter ring for $\Delta$



★ We will consider a list of elements $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_d)$ inside $\Bbbk[\Delta]$ where for each $0 \leq i \leq d$

$$\theta_i = \sum_{\substack{F \subset \Delta \\ dim(F)+1=i}} x_F.$$

$\boldsymbol{\theta}$ has these properties (by DeConcini, Eisenbud, and Procesi 1982):

1. $\boldsymbol{\theta}$ is algebraically independent, i.e., they generate a polynomial subalgebra, $\Bbbk[\boldsymbol{\theta}]$, of $\Bbbk[\Delta]$.
2. $\Bbbk[\Delta]$ can be finitely generated as a $\Bbbk[\boldsymbol{\theta}]$-module.

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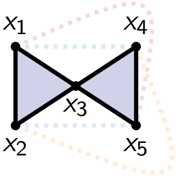
$\Delta =$ 


We first compute the Stanley-Reisner ring:

$$\Bbbk[\Delta] = \Bbbk[x_1, x_2, x_3, x_4, x_5]/\langle x_1x_4, x_1x_5, x_2x_4, x_2x_5\rangle.$$

Then our parameters are:

$$\theta = \begin{cases} \theta_1 = x_1 + x_2 + x_3 + x_4 + x_5 \\ \theta_2 = x_1x_2 + x_1x_3 + x_2x_3 + x_3x_4 + x_3x_5 + x_4x_5 \\ \theta_3 = x_1x_2x_3 + x_3x_4x_5 \end{cases}$$

Punchline: We may resolve $\Bbbk[\Delta]$ as a $\Bbbk[\theta]$-module.

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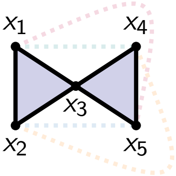
$\Delta =$ 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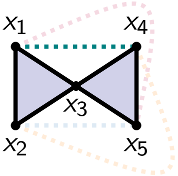
$\Delta =$ 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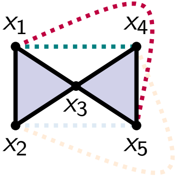
$\Delta =$ 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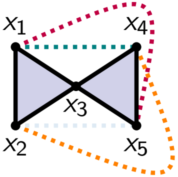
$\Delta =$ 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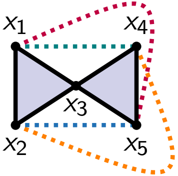
$\Delta =$ 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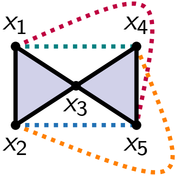
$\Delta =$ 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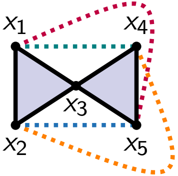
$\Delta =$ 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# Free Resolution of $\Bbbk[\Delta]$ as a $\Bbbk[\theta]$-module

★ If we resolve $\Bbbk[\Delta]$ (from the previous slide) as a $\Bbbk[\theta]$-module, then we obtain the following minimal free resolution:

$$\mathcal{F}_\bullet : \quad 0 \longleftarrow k[\Delta] \longleftarrow k[\theta]^1 \oplus k[\theta](-1)^4 \oplus k[\theta](-2)^5 \oplus k[\theta](-3)^3 \longleftarrow k[\theta](-5)^1 \longleftarrow 0$$

# Graded Betti Numbers

## Definition

Let $S$ be a polynomial ring and let $\mathcal{F}_\bullet$ be a minimal free resolution of a finitely generated $\mathbb{N}$-graded module $M$, and $F_i = \bigoplus_{j\in\mathbb{N}} S(-j)^{\beta_{i,j}}$, then the $i$-th **graded Betti number** of $M$ in degree $j$ is the invariant $\beta_{i,j}(M) = \beta_{i,j}$.

$$\begin{array}{ccccccccccc} \mathcal{F}_\bullet : & 0 & \longleftarrow & M & \xleftarrow{\partial_0} & F_0 & \xleftarrow{\partial_1} & \cdots & \xleftarrow{\partial_d} & F_d & \longleftarrow 0 \\ & & & & & \| & & & & \| & \\ & & & & & \bigoplus_{j\in\mathbb{N}} S(-j)^{\beta_{0,j}} & & & & \bigoplus_{j\in\mathbb{N}} S(-j)^{\beta_{d,j}} & \end{array}$$

## Note:

- The resolution $\mathcal{F}_\bullet$ is a **complex** meaning that $\partial_i \circ \partial_{i+1} = 0$ for all $0 \leq i \leq d$.
- The resolution $\mathcal{F}_\bullet$ is **exact** everywhere but in homological degree 0, meaning that for all $1 \leq i \leq n$, $\ker(\partial_i) = \mathrm{im}(\partial_{i+1})$.

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# Special Case: Cohen-Macaulay

## Definition

A simplicial complex $\Delta$ is **Cohen-Macaulay** if $\Bbbk[\Delta]$ is a free $\Bbbk[\theta]$-module, i.e., the resolution stops at $F_0$.

$$\mathcal{F}_\bullet: \quad 0 \longleftarrow \Bbbk[\Delta] \xleftarrow{\partial_0} F_0 \longleftarrow 0$$

$$F_0 = \bigoplus_{j\in\mathbb{N}} \Bbbk[\theta](-j)^{\beta_{0,j}}$$

## Theorem (A - Reiner; 2019)

*When $\Delta$ is Cohen-Macaulay, then $\beta_{0,j}$ of the free resolution of $\Bbbk[\Delta]$ as a $\Bbbk[\theta]$-module are completely determined by $f(\Delta) = (f_{-1}, f_0, \ldots, f_{d-1})$.*

Punchline: For $\Delta$ Cohen-Macaulay, the $\beta_{i,j}$ give no new information.

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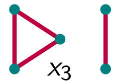
# Special Case: Graphs

**Theorem (A - Reiner; 2019)** Let $\Delta$ be a 1 dimensional simplicial complex with $f(\Delta) = (f_{-1}, f_0, f_1)$, then $\Bbbk[\Delta]$ has the following resolution as a $\Bbbk[\boldsymbol{\theta}]$-module:

$$0 \longleftarrow \Bbbk[\Delta] \longleftarrow \begin{array}{c} \Bbbk[\boldsymbol{\theta}]^1 \\ \bigoplus \\ \Bbbk[\boldsymbol{\theta}](-1)^{f_0 - 1} \\ \bigoplus \\ \Bbbk[\boldsymbol{\theta}](-2)^{f_1 - 1} \\ \bigoplus \\ \Bbbk[\boldsymbol{\theta}](-3)^{\tilde{\beta}_1(\Delta)} \end{array} \longleftarrow \Bbbk[\boldsymbol{\theta}](-3)^{\tilde{\beta}_0(\Delta)} \longleftarrow 0$$

where $\tilde{\beta}_i(\Delta)$ is the $i$-th reduced Betti number of $\Delta$. More specifically, $\tilde{\beta}_1(\Delta) = \#$ independent cycles and $\tilde{\beta}_0(\Delta) = \#$ connected components $-1$.

# Graph Example

$\Delta =$ 


$$f(\Delta) = \begin{matrix} ( & f_{-1}, & f_0, & f_1 & ) \\ ( & 1, & 5, & 4 & ) \end{matrix}$$

$$\tilde{\beta}_1(\Delta) = 1, \quad \tilde{\beta}_0(\Delta) = 1$$

$$0 \longleftarrow \Bbbk[\Delta] \longleftarrow \Bbbk[\theta]^1 \longleftarrow \Bbbk[\theta](-3)^1 \longleftarrow 0$$

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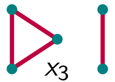
$\Delta =$ 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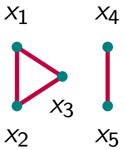
$\Delta =$ 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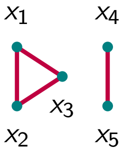


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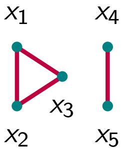
$$k[\theta](-2)^3$$

$$\oplus$$

$$k[\theta](-3)^1$$

# Graph Example

$\Delta =$ 


$$\begin{matrix} & ( & f_{-1}, & f_0, & f_1 & ) \\ f(\Delta) = & ( & 1, & 5, & 4 & ) \\ & & \tilde{\beta}_1(\Delta) = 1, & \tilde{\beta}_0(\Delta) = 1 & & \end{matrix}$$

$$0 \longleftarrow \Bbbk[\Delta] \longleftarrow \Bbbk[\theta]^1 \longleftarrow \Bbbk[\theta](-3)^1 \longleftarrow 0$$

$$\bigoplus$$

$$k[\theta](-1)^4$$

$$\bigoplus$$

$$k[\theta](-2)^3$$

$$\bigoplus$$

$$k[\theta](-3)^1$$

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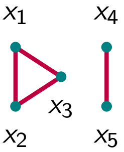
$\Delta =$ 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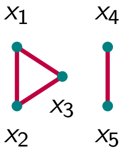
$\Delta =$ 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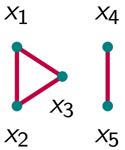
$\Delta =$ 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$$\bigoplus$$

$$k[\theta](-2)^3$$

$$\bigoplus$$

$$k[\theta](-3)^1$$

# Special Case: Graph Corollary

## Corollary (A - Reiner; 2019)

For any simplicial cell complex $\Delta$ of dimension 1, with vertex set $V(\Delta)$ and edge set $E(\Delta)$, $\mathbb{k}[\Delta]$ is isomorphic as a graded (virtual) $\mathrm{Aut}(\Delta)$-representation to

$$\mathbb{k}[\boldsymbol{\theta}] \otimes \Big( \mathbb{k}(0) \\ \oplus \mathbb{k}[V(\Delta)]/\mathbb{k}\text{-span}\{\theta_1\}(-1) \\ \oplus \mathbb{k}[E(\Delta)]/\mathbb{k}\text{-span}\{\theta_2\}(-2) \\ \oplus \Big(\tilde{H}^1(\Delta) - \tilde{H}^0(\Delta)\Big)(-3)\Big)$$

# Overview

# Future Work & Cohen-Macaulay Simplicial Posets



**Conjecture (A - Reiner; 2019)** Let $\Delta$ be a simplicial cell complex of dimension $d-1$ with barycentric subdivision $Sd(\Delta)$, and let $\mathrm{Aut}(\Delta)$ be the group of automorphisms of $\Delta$. Then

$$\mathrm{Tor}_i^{\Bbbk[\theta]}(\Bbbk[\Delta], \Bbbk)_j \cong \bigoplus_{S \subseteq [d] : \sum_{s \in S} s = j} \tilde{H}^{|S|-i-1}(Sd(\Delta)|_S)$$

as (virtual) $\mathrm{Aut}(\Delta)$-representations.

## Theorem (A - Reiner; 2019)

*Let $P$ be a Cohen-Macaulay simplicial poset and let $G$ be a group of poset automorphisms of $P$ that extends to ring automorphims of $\Bbbk[P]$. Then, for all $i \geq 0$, the $i$-th graded piece of $\Bbbk[P]/(\theta)$ is isomorphic to the direct sum of irreducible characters of $\tilde{H}_{|S|-i+1}(P_S)$, where $P_S$ is a rank-selected poset of $P$, for all $S \subseteq [n]$ with $\sum_{s \in S} s = i$.*

# Future Work

## Question

Given a simplicial cell complex $\Delta$ or a simplicial poset $P$, is the depth of $\Bbbk[\Delta]$ or $\Bbbk[P]$ as a $\Bbbk[\boldsymbol{\theta}]$-module the same as the max $m > 0$ such that $(\theta_1, \ldots, \theta_m)$ is a regular sequence acting on $\Bbbk[\Delta]$ or acting on $\Bbbk[P]$?

★ If yes, this would be analogous to a conjecture by Landweber and Stong (1987).

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★ If yes, this would be analogous to a conjecture by Landweber and Stong (1987).

# Overview

# References

$$T + Ht + At^2 + Nt^3 + Kt^4 + St^5$$


1. You can find these slides on: ashleigh-adams.com
2. *Hodge Algebras*, Corrando De Concini, David Eisenbud, and Claudio Procesi
3. *Simplicial Posets- $f$-vectors and Free Resolutions*, Arthur Duval
4. *Some Aspects of Groups Acting on Finite Posets*, Richard Stanley
5. *$f$-vectors and $h$-vectors of simplicial posets*, Richard Stanley
6. *The depth of rings of invariants over finite fields*, Peter Landweber and Robert Stong.

# A Canonical H.S.O.P. for $\Bbbk[\Delta]$

**Theorem (De Concini, Eisenbud, Procesi 1982)**

*Let $R$ be a noetherian ring and let $A$ be a commutative $\Bbbk$-algebra generated by $V \subset A$. If $V = \bigcup_{i=0}^{n} V_i$, where each $V_i$ is a clutter, then $VA$ is nilpotent modulo the ideal generated by the elements $\theta_i = \sum_{x \in V_i} x$ for all $i \in [n]$ and $A/(\boldsymbol{\theta})A$ is generated as an $R$-module by square-free standard monomials.*

Let $P$ be a simplicial poset of rank $n$, and let $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$ be a sequence such that for each $0 \leq i \leq n$

$$\theta_i = \sum_{\substack{F \subset P \\ rank(F) = i}} x_F .$$

Since each term in $\theta_i$ is determined by incomparable faces in $P$, then $P$ forms an H.S.O.P. for $\Bbbk[P]$.

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# Representations of $G$ on $P_S$



Let $G$ be a group acting on the simplicial poset $P$. Since every element in $G$ that acts on $P_S \in P$ is order preserving, then $G$ also acts on the on the reduced homology group $\tilde{H}_i(P_S)$ for $-1 \leq i \leq |S| - 1$. Let

$$\gamma_{S,i} : G \to Aut(\tilde{H}_i(P_S)),$$

then we define a virtual representation of $G$ to be

$$\beta_S := \sum_{i \geq 1} (-1)^{|S|-1-i} \gamma_{S,i}.$$

**Lemma (Stanley 1981)**

*Let $S = \{s_1 < s_2 < \cdots < s_m\} \subseteq [n]$ and let $P_S$ denote a rank-selected subposet of a poset $P$. If $P$ is Cohen-Macaulay, then $\beta_S$ is isomorphic to $\chi_{\tilde{H}_{|S|-1}(P_S)}$.*