FURTHER WORK ON A CONJECTURE ON STANLEY-REISNER RINGS

By

ASHLEIGH N. ADAMS
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Approved:

Erik Carlsson, Chair

Eugene Gorsky

Victor Reiner

Committee in Charge

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To my younger self.

Contents

Abstract

Given a simplicial complex Δ and its barycentric subdivision $\mathrm{Sd}\Delta$, we explore two homogeneous systems of parameters: one is the elementary symmetric functions inside the $\mathbb{N}$-graded Stanley-Reisner ring of Δ, $\Bbbk[\Delta]$, and the other is written based on a balanced coloring of $\mathrm{Sd}\Delta$ which lives inside the $\mathbb{N}^d$-graded Stanley-Reisner ring of $\mathrm{Sd}\Delta$, namely, $\Bbbk[\mathrm{Sd}\Delta]$. The first is stable under symmetries and the other is stable under colorful automorphisms of $\mathrm{Sd}\Delta$. In this paper, we develop methodology to explore comparing the resolutions of $\Bbbk[\Delta]$ and $\Bbbk[\mathrm{Sd}\Delta]$ over their respective parameter rings. We then prove it in the trivial case, which is simply the coinvariant algebra with alternate grading and one non-trivial case.

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CHAPTER 1

Introduction

Given a polynomial ring $S = \Bbbk[x_1, \ldots, x_n]$ any squarefree monomial ideal is a Stanley-Reisner ideal denoted I_Δ. Those monomials which generate I_Δ define non-faces of a simplicial complex. This bijection between homogeneous squarefree monomial ideals and (regular,finite) simplicial complexes leads us to be able to answer questions about the topology of Δ through study of the related Stanley-Reisner ring $\Bbbk[\Delta] = S/I_\Delta$. In [1], we investigated nice properties of a certain homogeneous system of parameters, called the universal system of parameters by Herzog and Moradi [12]. The universal system of parameters, $\theta = (\theta_1, \ldots, \theta_d)$, are the universal system of parameters in the image of the quotient map $S \to S/I_\Delta$. These are particularly nice since they are stable under symmetries. This ring is standard $\mathbb{N}$-graded, where $\deg_{\mathbb{N}}(x_i) = 1$.

We then studied a particular class of simplicial complexes, namely, balanced simplicial complexes. Given a simplicial complex Δ on a vertex set V, a *proper coloring* is a map $\kappa : V \to [\ell]$ such that for all edges $\{i, j\}$ in Δ, $\kappa(i) \neq \kappa(j)$. A simplicial complex of dimension $d-1$ is called balanced if there is a proper coloring with precisely d colors. In this thesis we are particularly interested in studying a certain class of balanced simplicial complexes, namely, barycentric subdivisions of simplicial complexes, which will be denoted SdΔ. We may define a homogeneous system of parameters, $\gamma = (\gamma_1, \ldots, \gamma_d)$ where γ_i is the sum of all vertices of color i. These systems of parameters are particularly nice, since they are stable under the colored automorphims of Δ. This ring is $\mathbb{N}^d$-graded where $\deg_{\mathbb{N}^d}(y_F) = \epsilon_{\kappa(F)}$, where $\epsilon_{\kappa(F)}$ is the standard basis vector in $\mathbb{N}^d$.

Let $A = \Bbbk[z_1, \ldots, z_d]$ and consider the two maps:

$$A \to \Bbbk[\Delta]$$

$$z_j \mapsto \theta_j$$

and

$$A \to \Bbbk[\mathrm{Sd}\Delta]$$
$$z_j \mapsto \gamma_j$$

For the second map, we apply a grading specialization stated in Definition 3.1.5, $\mathbb{N}^d \to \mathbb{N}$, taking $\epsilon_j \mapsto j$. The paper [1] asks:

Is the shape of the (minimal, finite, free) resolution of $\Bbbk[\Delta]$ over A is the same as the resolution of $\Bbbk[\mathrm{Sd}\Delta]$ over A (under the grading specialization).

Since two rings are isomorphic as A-modules if their resolutions over A are the same, then showing the resolutions are equal gives us Conjecture 3.3.4. If $\mathcal{F}_\bullet^\Delta$ is a resolution of $\Bbbk[\Delta]$ over A and $\mathcal{F}_\bullet^{\mathrm{Sd}\Delta}$ is a resolution of $\Bbbk[\mathrm{Sd}\Delta]$ over A, then a weaker version of Conjecture 3.3.4, is Conjecture 3.3.2, which asks if the graded Betti numbers of $\mathcal{F}_\bullet^\Delta$ are the same as the graded Betti numbers of $\mathcal{F}_\bullet^{\mathrm{Sd}\Delta}$. In this thesis, we attempt to extend the study of these resolutions by setting up the theory for a new approach to resolving Conjectures 3.3.2 or 3.3.4 and proving it for both the trivial case and a non-trivial base case.

Our set up is as follows. We observe that

$$\mathrm{Tor}_0^A(\Bbbk[\Delta], \Bbbk) \cong \Bbbk[\Delta]/(\theta_1, \ldots, \theta_d).$$

However, when Δ is an n-simplex, I_Δ is empty and $\Bbbk[\Delta] = S = \Bbbk[x_1, \ldots, x_n]$. Thus

$$\mathrm{Tor}_0^A(\Bbbk[\Delta], \Bbbk) \cong S/(\theta_1, \ldots, \theta_n) = S/(e_1(\mathbf{x}), \ldots, e_n(\mathbf{x}))$$

where $e_1(\mathbf{x}), \ldots, e_n(\mathbf{x})$ are the elementary symmetric functions in variables $\mathbf{x} = (x_1, \ldots, x_n)$. Let I_n be the ideal generated by the n elementary symmetric functions in variables $\mathbf{x}$. This ring,

$$\mathcal{A}_n := S/(e_1(\mathbf{x}), \ldots, e_n(\mathbf{x})),$$

is simply the coinvariant algebra, which is a well studied object in combinatorics, algebraic geometry, and representation theory. If I_Δ is non-empty, then we define $I_n^\Delta = I_n + I_\Delta$. The ring

$$\mathcal{A}_n^\Delta := S/I_n^\Delta$$

is a generalized coinvariant algebra. It is our goal in much of the thesis to understand this ring. In particular, we give an explicit conjectural basis for when I_Δ is generated by a single monomial. We also compute the Hilbert series of $\mathcal{A}_n^\Delta$ and give a combinatorial method for computing the Hilbert series. We believe that this should extend naturally with a bit more time.

The thesis is structured as follows. In Chapter 2 we layout the preliminaries for simplicial complexes, Stanley-Reisner rings (Section 2.1), Hilbert series (Section 2.2), resolutions of Stanley-Reisner rings (Section 2.3), and universal system of parameters (Section 2.4), all of which will be required for the rest of the thesis.

In Chapter 3 we focus on introducing in detail the motivating conjectures for this work. In Section 3.1 we give a presentation of the Stanley-Reisner ring of the barycentric subdivision of a simplicial complex and introduce a certain system of parameters for this ring. In Section 3.2, we review a generalization of a well-known formula of Hochster. This all leads to the two motivating conjectures which are stated in Section 3.3.

In Chapter 4, we review monomial orderings (Section 4.1), Gröbner bases and Buchbergers algorithm (Section 4.2). We then review symmetric polynomials (Section 4.3.1) with the purpose of introducing the coinvariant algebra in the same section. We then introduce a combinatorial method for computing the Artin basis for the coinvariant algebra (Section 4.3.3).

In Chapter 5, we discuss a generalization of the coinvariant algebra (Section 5.1), give its Hilbert series and generalize the combinatorial model from the previous section for a certain case. We then give an explicit (conjectural) description of the basis of the generalized coinvariant algebra for a certain case (Section 5.2). We also give the code used in computational verification (Section 5.3).

In Chapter 6, we set up the necessary machinery to use the methodology constructed in the previous chapters to eventually be able to resolve one of the motivating conjectures.

CHAPTER 2

Stanley-Reisner rings

2.1. Simplicial complexes & their Stanley-Reisner rings

Let $S = \Bbbk[x_1, \ldots, x_n]$ be a polynomial ring. Using multi-index notation, a *monomial* $\mathbf{x}^{\mathbf{a}} := x_1^{a_1} \cdots x_n^{a_n}$ for a vector $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$. The support set of $\text{supp}(\mathbf{a}) = \{i : a_i > 0\}$. Then, we use the notation that for $F \subset [n]$, $\mathbf{x}^F$ is the monomial $\mathbf{x}^{\mathbf{a}}$ with $\text{supp}(\mathbf{a}) = \{i : a_i = 1 \text{ if } i \in F\}$. In other words, $\mathbf{x}^F = \prod_{i \in F} x_i$. A monomial is called *squarefree* if for all $a_i \in \alpha$, $a_i = 0$ or $a_i = 1$. An ideal is squarefree if it is generated by squarefree monomials. Let I_Δ be an ideal generated by homogeneous squarefree monomials. This ideal is called the Stanley-Reisner ideal and the quotient ring S/I_Δ is called the Stanley-Reisner ring. This ring has an alternate topological characterization as first laid out in Reisner's P.h.D. thesis at the University of Minnesota, Twin Cities. CITATION In this section we give a general introduction to simplicial complexes and Stanley-Reisner theory. For more a more in depth exploration of the bijection between squarefree monomial ideals and simplicial complexes see [**13**].

Definition 2.1.1. *A Δ be a (finite, regular) simplicial complex on n vertices is a collection of subsets $G \subseteq [n] := \{1, \ldots, n\}$, called* faces*, such that the following hold*

- *Each $v \in [n]$ is also in Δ;*
- *If $F' \subseteq F$, and $F \in \Delta$, then $F' \in \Delta$.*

We note that $\varnothing$ is always in Δ. A face F if called a *facet* if there does not exist $F' \in \Delta$ such that $F \subseteq F'$. The *dimension* of a face F in Δ, denoted $\dim(F) = |F| - 1$. It follows that if F is a maximal facet, i.e, $\dim(F) \geq \dim(F_i)$ for all other facets $F_1, \ldots, F_k$ of Δ, the dimension of Δ, $\dim(\Delta) = \dim(F) = |F| - 1$. A simplicial complex is completely determined by its facets. A simplicial complex $\Delta = 2^{[n]}$ is called an n-simplex and is denoted Δ_n and has dimension $n-1$. Since each facet is a simplex, one can alternatively define $\Delta = \bigcup_{1 \leq k \leq n} F_i$ for simplicies F_i on a vertex set $[n]$. A *minimal non-face* of Δ is a subset $G \subseteq [n]$ such that $G \notin \Delta$ (non-face) and there does not exist any other subset $G' \subseteq [n]$ such that $G' \subseteq G$ (minimality). One may also completely characterize a simplicial complex on n vertices by its non-facets by removing the minimal-non faces from the simplex Δ_n in such a way that it respects the simplicial complex structure.

A *subcomplex* Δ' of Δ is a collection of faces of Δ which respects the simplcial complex requirements. We define three particularly important subcomplexes of Δ

- The k-*skeleton* of Δ is the complex $\Delta^{(k)} := \{F \in \Delta : \dim(F) \leq k\}$.
- The *star* of a face $F \in \Delta$ is $\mathrm{st}(F) := \{F' \in \Delta : F \subseteq F'\}$.
- The *closed star* $\overline{\mathrm{st}(F)}$ is the smallest subcomplex of Δ that contained $\mathrm{st}(F)$.
- The *link* of a face $F \in \Delta$ is $\mathrm{link}(F) := \{F' \in \overline{\mathrm{st}(F)} : F \cap F' = \varnothing\}$.
- A *sub-restricted simplicial complex* $\Delta|_S$, with $S \subset [n]$, is generated by all faces $F \in \Delta$ such that $F \subseteq S$.

We may encode topological data of the simplicial complex in an algebraic structure. This algebraic structure, called the Stanley-Reisner ring, is the focus of this section and is a particular case of a broader class of algebraic objects called *face rings*, which encode topological data (in particular homological data) of cell complexes.

Definition 2.1.2. *Let Δ be a simplicial complex on n vertices and let $S = \Bbbk[x_1, \ldots, x_n]$, where $\Bbbk$ is a field. We define the* Stanley-Reisner ideal $I_\Delta = (x^G : G \notin \Delta)$. *By the Hilbert Basis theorem, it is sufficient to say that I_Δ is generated by all minimal non-faces of Δ and this set is finite. The* Stanley-Reisner ring *is the quotient* $\Bbbk[\Delta] := S/I_\Delta$.

The ring $\Bbbk[\Delta]$ has a particularly nice S-module structure. Not only is $\Bbbk[\Delta]$ an $\mathbb{N}^n$-graded (or $\mathbb{Z}$-graded for a coarser grading) finitely generated S-module, but its Krull dimension is exactly $d = \dim(\Delta) + 1$.

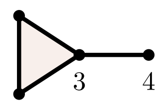
Definition 2.1.3. *Given a commutative ring R, an R-module M is $\mathbb{N}^n$-graded if*

$$M = \bigoplus_{\mathbf{b} \in \mathbb{N}^n} M_{\mathbf{b}}$$

and for $\mathbf{x^a} \in R$ $\mathbf{x^a} M_{\mathbf{b}} \subseteq M_{\mathbf{a+b}}$.

Since $\Bbbk[\Delta]$ is a finitely generated $\mathbb{N}^n$ graded S-module it is natural to ask about writing down a (finite, free) resolution for $\Bbbk[\Delta]$. We will address this shortly, but before proceeding, we will give a classic example.

Example 2.1.4. *Let $n = 4$ and let Δ be the simplicial complex given by facets $F_1 = \{1, 2, 3\}$ and $F_2 = \{3, 4\}$. The dimension of F_1 is 2 and the dimension of F_2 is 1. The complex has the graphical form:*

$\Delta =$ 

The Stanley-Reisner ring for Δ is

$$\Bbbk[\Delta] = \Bbbk[x_1, x_2, x_3, x_4]/(x_1x_4, x_2x_4).$$

The ideal here is given by the two minimal non-faces in Δ, that is $\{1, 4\}$ and $\{2, 4\}$.

2.2. f-vectors, h-vectors, and Hilbert Series

Combinatorially, we can keep track of the number of faces of each dimension in a simplicial complex of dimension d through the *f-vector* $\mathbf{f} := (f_{-1}, f_0, \ldots, f_d)$ where $f_i = \#\{F \in \Delta : \dim(F) = i\}$. Although elementary in nature, we may obtain homological information via the f-vector via the relation:

$$\sum_{i=0}^{d} f_{i-1}(q-1)^{d-i} = \sum_{j=0}^{d} h_j q^{d-j} \tag{2.1}$$

Thus the h-vector is uniquely determined by the f-vector. How does the h-vector give us anything interesting if it is so easily determined by the f-vector? To explain, we first give the definition of a generating function for $\Bbbk[\Delta]$.

Definition 2.2.1 (Hilbert Series for the Stanley-Reisner Ring). *We give three definitions for the* Hilbert series *of $\Bbbk[\Delta]$, each of which illuminates slightly different information:*

(1)

$$\text{Hilb}(\Bbbk[\Delta]; q) = \frac{h_0 + h_1 q + \cdots + h_d q^d}{(1-q)^d},$$

(2)

$$\mathrm{Hilb}(\Bbbk[\Delta]; q_1, \ldots, q_n) = \sum_{\mathbf{b}\in\mathbb{N}^n} \dim_{\Bbbk}(\Bbbk[\Delta]_{\mathbf{b}})\mathbf{q}^{\mathbf{b}}$$

(3)

$$\mathrm{Hilb}(\Bbbk[\Delta]; x_1, \ldots, x_n) = \sum_{F\in\Delta}\prod_{i\in F} \frac{x_i}{1-x_i}.$$

Remark 2.2.2. *Definition (3) may become definition (1) by setting $x_i = q$ for all $1 \le i \le n$.*

Example 2.2.3. *Referring to Example 2.1.4 we will compute its Hilbert series using Definition 2.2.1 (1) and (3).*

Using (1): *The f-vector of Δ is $\mathbf{f} = (1, 4, 4, 1)$ since we have one empty set, 4 vertices, 4 edges, and 1 2-face. Using the relations from Equation 2.1,*

$$1 \cdot (1-q)^4 + 4 \cdot (1-q)^3 + 4 \cdot (1-q)^2 + 1 \cdot (1-q) = 1 - 2q^2 + q^3.$$

Since $\dim(\Delta) = 3 - 1$, *then by (i) we have*

$$\mathrm{Hilb}(\Bbbk[\Delta]; q) = \frac{1-2q^2+q^3}{(1-q)^3}.$$

Using (2):

$$\begin{aligned}\mathrm{Hilb}(\Bbbk[\Delta]; x_1, \ldots, x_n) = 1 &+ \frac{x_1}{1-x_1} + \frac{x_2}{1-x_2} + \frac{x_3}{1-x_3} + \frac{x_4}{1-x_4}\\ &+ \frac{x_1}{1-x_1}\cdot\frac{x_2}{1-x_2} + \frac{x_1}{1-x_1}\cdot\frac{x_3}{1-x_3} + \frac{x_2}{1-x_2}\cdot\frac{x_3}{1-x_3} + \frac{x_3}{1-x_3}\cdot\frac{x_4}{1-x_4}\\ &+ \frac{x_1}{1-x_1}\cdot\frac{x_2}{1-x_2}\cdot\frac{x_3}{1-x_3}\\ &= \frac{x_1}{1-x_1} + \frac{x_2}{1-x_2} + \frac{x_3}{1-x_3} + \frac{x_4}{1-x_4}\\ &+ \frac{x_1x_2}{(1-x_1)(1-x_2)} + \frac{x_1x_3}{(1-x_1)(1-x_3)} + \frac{x_2x_3}{(1-x_2)(1-x_3)} + \frac{x_3x_4}{(1-x_3)(1-x_4)}\\ &+ \frac{x_1x_2x_3}{(1-x_1)(1-x_2)(1-x_3)}\end{aligned}$$

2.3. Resolutions of Stanley-Reisner rings

Given a commutative ring S and an S-module M it is natural to ask about the structure of M. Since bases for modules only exist when the module is free, then we must ask for a minimal system of generators. However, we are then faced with the problem that these generators do not give much information about the structure of M. We therefore use free resolutions to give an approximation of our module. In this section we will go over the basic theory about free resolutions. For more in depth descriptions see [7,13,15] and [6]. For more on the topic through the lens of toric topology, see [4].

Definition 2.3.1. *Given a commutative graded ring S and a finitely generated $\mathbb{N}^n$-graded S-module M, the sequence*

$$\begin{array}{ccccccccccccc} \mathcal{F}_\bullet : & 0 \longleftarrow M & \xleftarrow{\partial_0} & F_0 & \xleftarrow{\partial_1} & \cdots & \xleftarrow{\partial_d} & F_d & \longleftarrow & F_{d+1} & \longleftarrow & \cdots \\ & & & \| & & & & \| & & & & \\ & & & \bigoplus_{\mathbf{a}\in\mathbb{N}^n} S(-\mathbf{a})^{\beta_{0,\mathbf{a}}} & & & & \bigoplus_{\mathbf{a}\in\mathbb{N}^n} S(-\mathbf{a})^{\beta_{d,\mathbf{a}}} & & & & \end{array}$$

is a finite free minimal resolution of M if

(1) *$\mathcal{F}_\bullet$ is a* complex *meaning that $\partial_i \circ \partial_{i+1} = 0$ for all $0 \leq i \leq d$;*
(2) *$\mathcal{F}_\bullet$ is* exact *everywhere but in homological degree 0, meaning that for all $1 \leq i \leq n$, $\ker(\partial_i) = \mathrm{im}(\partial_{i+1})$;*
(3) *$\mathcal{F}_\bullet$ is* finite*, meaning that there exists d such that $F_i = 0$ for all $i > d$. This d is called the* length *of the resolution,*
(4) *$\mathcal{F}_\bullet$ is* minimal*, meaning that $\partial_{i+1}(F_{i+1}) \subseteq R_+F_i$ where R_+ is the irrelevant ideal, i.e., no invertible elements appear in the differential matrices.*

Remark 2.3.2. *In this paper we are only dealing with finite free minimal resolutions. However, there are many modules that are not finitely generated. These modules still have free resolutions to give an approximation of the structure of the module. In this case, we drop the finite requirement. Furthermore, finite free resolutions need not be minimal. Not only can finite free resolution may be refined to be minimal, every minimal resolution for an S-module M is isomorphic.*

As referenced earlier, when $\Bbbk[\Delta] = S/I_\Delta$ is viewed as an S-module, it has Krull dimension $d = \dim(\Delta) + 1$. For general S-modules, this is not always so concrete. We are able to come to this conclusion due to the Hilbert Syzygy theorem.

Theorem 2.3.3 (Hilbert's Syzygy Theorem). *If M is a finitely generated module over a polynomial ring $\Bbbk[x_1, \ldots, x_n]$, then M has a finite free resolution of length at most n.*

Another way to phrase the Hilbert Syzygy theorem is that the F_i in a resolution $\mathcal{F}_\bullet$ of an S-module M are always free modules over S. More importantly, the Hilbert Syzygy theorem tells us that the length of the resolution is not only finite, but its length is bounded by n.

In the definition of free resolutions, Definition 2.3.1, we see that $F_i = \bigoplus_{\mathbf{a}\in\mathbb{N}^n} S(-\mathbf{a})^{\beta_{i,\mathbf{a}}}$, which is called the *i-th syzygy module.* We will now dissect this definition. Here, $S(-\mathbf{a})$ is the free module generated in degree $\mathbf{a} \in \mathbb{N}^n$ and is isomorphic to $(\mathbf{x}^{\mathbf{a}})$ as an $\mathbb{N}^n$-graded S-module. Each $S(-\mathbf{a})$ has a given rank denoted $\beta_{i,\mathbf{a}} = \beta_{i,\mathbf{a}}(M)$. This invariant is called the *i-th Betti number* of M, and measures the minimal number of generators in degree $\mathbf{a}$ for F_i. The Betti numbers are a topic of great interest in various fields of mathematics. They have many definitions, however, we will use the functorial definition in this paper.

Definition 2.3.4 (Betti numbers). *Let $S = \Bbbk[x_1, \dots, x_n]$ be a polynomial ring and let M be an S-module. For $\mathbf{a} \in \mathbb{N}^n$, the i*-th Betti number

$$\beta_{i,\mathbf{a}} = \dim_{\Bbbk} \operatorname{Tor}_i^S(M, \Bbbk)_{\mathbf{a}}.$$

It is often an area of interest to compute the Betti numbers given a module since these are not always known (or simple) to compute. However, in our setting the Betti numbers can be explicitly computed via Hochster's formula. In order to state Hochster's formula for computing the Betti numbers for a simplicial complex, we first translate $\mathbb{N}^n$-graded Betti numbers for I_Δ into $\mathbb{N}$-graded Betti numbers for S/I_Δ. The equalities line up rather simply so that shifting to the quotient from the ideal only shifts the indexing of the Betti numbers by 1:

$$\beta_{i,j}(I_\Delta) = \sum_{|\mathbf{a}|=j} \beta_{i,\mathbf{a}}(I_\Delta) = \sum_{|\mathbf{a}|=j} \beta_{i+1,\mathbf{a}}(S/I_\Delta) = \beta_{i+1,j}(S/I_\Delta).$$

Theorem 2.3.5 (Hochster's Formula, [10]). *For any simplicial complex Δ, given $\Bbbk[\Delta] = S/I_\Delta$ as an S-module,*

$$\beta_{i,j}(\Bbbk[\Delta]) = \sum_{\substack{|\mathbf{a}|=j,\\ \mathbf{a}\in\{0,1\}^n}} \dim_{\Bbbk} \tilde{H}^{j-i+1}(\Delta|_{\mathbf{a}}, \Bbbk).$$

Example 2.3.6. *Let Δ be the simplicial complex from Example 2.1.4. It has resolution*

$$\mathcal{F}_\bullet: \quad 0 \longleftarrow \Bbbk[\Delta] \xleftarrow{\partial_0} \underset{\underset{S(-0)^1}{\parallel}}{F_0} \xleftarrow{\partial_1} \underset{\underset{S(-1)^2}{\parallel}}{F_1} \xleftarrow{\partial_2} \underset{\underset{S(-3)^1}{\parallel}}{F_2} \longleftarrow 0$$

The differentials are

$$\partial_0 = \begin{bmatrix} x_1x_4 & x_2x_4 \end{bmatrix}, \quad \partial_1 = \begin{bmatrix} -x_2 \\ \\ x_1 \end{bmatrix}, \quad \partial_2 = 0.$$

2.4. Systems of parameters and R-sequences for $\Bbbk[\Delta]$

Definition 2.4.1. *Let $S = \Bbbk[x_1, \ldots, x_n]$ and let M be an S-module with Krull dimension k. Then $f_1, \ldots, f_k \in M$ is a* system of parameters (SOP) *if the length of $M/(f_1, \ldots, f_k)M$ is finite. A* homogeneous system of parameters (HSOP) *is a SOP which is homogeneous. A* linear system of parameters (LSOP) *is a system of parameters which is linear.*

Definition 2.4.2 (Universal System of Parameters (i)). *Let Δ be a simplicial complex on n vertices with Stanley-Reisner ring $\Bbbk[\Delta] = S/I_\Delta$. The universal system of parameters are the elementary symmetric functions under the image of the quotient map $S \to S/I_\Delta$.*

Alternatively, may also define the universal system of parameters via the faces of the simplicial complex.

Definition 2.4.3 (Universal System of Parameters (ii)). *Let Δ be a simplicial complex on n vertices with dimension $d-1$ and with Stanley-Reisner ring $\Bbbk[\Delta] = S/I_\Delta$. The universal system of parameters is defined to be the sequence $\Theta = (\theta_1, \ldots, \theta_d)$ where*

$$\theta_i = \sum_{\substack{F \in \Delta \\ |F|=i}} x_i.$$

The universal system of parameters have shown up in various places before being so named by Herzog and Moradi [12]. Work by De Concini, Eisenbud, and Procesi [5] on algebras with straightening laws; Garsia and Stanton [9] on their work in invariant theory of permutation groups; and finally, D.E. Smith [14] in his algebraic geometric work on sheaves of posets.

Example 2.4.4. *Computing the universal system of parameters for the Stanley-Reisner ring in Example 2.1.4, we have* $\Theta = (\theta_1, \theta_2, \theta_3)$ *where*

$$\theta_1 = \underbrace{x_1 + x_2 + x_3}_{vertices} \quad \theta_2 = \underbrace{x_1x_2 + x_1x_3 + x_2x_3 + x_3x_4}_{edges} \quad \theta_3 = \underbrace{x_1x_2x_3}_{the\ 2-face}$$

Since the dimension of Δ *is* 2*, then the length of this sequence is* $3 = \dim(\Delta) + 1$.

One special property of an HSOP, which is sometimes used as the definition of an HSOP is this: If $f_1, \ldots, f_k$ is an HSOP for M, then they generate a finitely generated algebra $A = \Bbbk[f_1, \ldots, f_k]$ over which M is a finitely generated module. So we may construct a parameter ring from any HSOP. Consider the map $\Bbbk[\Delta] \to A = \Bbbk[z_1, \ldots, z_d]$ with $\theta_i \mapsto z_i$. We may write $\Bbbk[\Delta]$ as a finitely generated graded A-module. It is the focus of this paper to understand that module structure.

CHAPTER 3

Conjectures on Stanley-Reisner rings

3.1. Stanley-Reisner rings of barycentric subdivisions

The barycentric subdivision of a simplicial complex is a simplicial complex of the same dimension, and thus has a Stanley-Reisner ring in the sense of Chapter 2. However, for our purposes, we give a slightly non-standard labeling of the variables. Recall that in this paper we are using multi-index notation so that a monomial $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$, but that $\mathbf{x}^G = \prod_{i \in G} x_i$. Another way of indexing variables is through sets so that for $S = \{i_1, \ldots, i_k\} \subset [n]$, $y_S = y_{i_1, \cdots, i_k}$. Then if $S_1, \ldots, S_k \subseteq [n]$, a monomial in this setting would be of the form $\mathbf{y}^{\mathbf{a}} = y_{S_1}^{a_1} \cdots y_{S_k}^{a_k}$.

Definition 3.1.1 (Stanley-Reisner ring of the Barycentric Subdivision of Δ). *Let Δ be a simplicial complex on vertex set $[n]$ of dimension $d-1$, and let $\mathrm{Sd}\Delta$ be its baryentric subdivision. Then we define the Stanley-Reisner ring for $\mathrm{Sd}\Delta$ to be*

$$\Bbbk[\mathrm{Sd}\Delta] := \Bbbk[y_F : F \in \Delta]/I_{\mathrm{Sd}\Delta}$$

where

$$I_{\mathrm{Sd}\Delta} = (y_F y_G : F \nsubseteq G \text{ or } G \nsubseteq F).$$

Note that $I_{\mathrm{Sd}\Delta}$ is given by the standard Stanley-Reisner relations. Furthermore, this ring has the quality that if $\mathbf{y}^{\mathbf{a}} \in \Bbbk[\mathrm{Sd}\Delta]$, then

$$\mathbf{y}^{\mathbf{a}} = y_{F_1}^{a_1} y_{F_2}^{a_2} \cdots y_{F_k}^{a_k},$$

then $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_k \subseteq [n]$, i.e., the F_i are a nested chain. There are two natural gradings one may give $\Bbbk[\mathrm{Sd}\Delta]$, one is an $\mathbb{N}$-grading, and the other is an $\mathbb{N}^d$-grading.

Remark 3.1.2. *Although, this particular definition was arrived at independently, it is also used by Huang in [11] and is written in terms of Boolean algebras. In his case, it is written in terms of the face poset of Δ. In terms of poset language, $I_{\mathrm{Sd}}\Delta$ is generated by all $y_{G_1} \cdots y_{G_k} \in \Delta$ such that $G_1, \ldots, G_k$ are incomparable in the face poset of Δ.*

Definition 3.1.3 ($\mathbb{N}$-grading as a ring). *As a ring,* $\Bbbk[\mathrm{Sd}\Delta]$ *may be given the standard* $\mathbb{N}$*-grading where*

$$\deg_{\mathbb{N}} y_F := \dim_\Delta F + 1.$$

Definition 3.1.4 ($\mathbb{N}^d$-grading as a vector space). *It is natural to give the ring* $\Bbbk[\mathrm{Sd}\Delta]$ *a multigrading by* $\mathbb{N}^d$ *where*

$$\deg_{\mathbb{N}^d}(y_F) := \epsilon_i,$$

$F \in \Delta$ *when* $\dim_\Delta F = i - 1$ *and* ϵ_i *is the* i*-th standard basis vector. So then for all vertices* $v \in \Delta$ *the elements* y_v *in* $\Bbbk[\mathrm{Sd}\Delta]$ *have degree* ϵ_1.

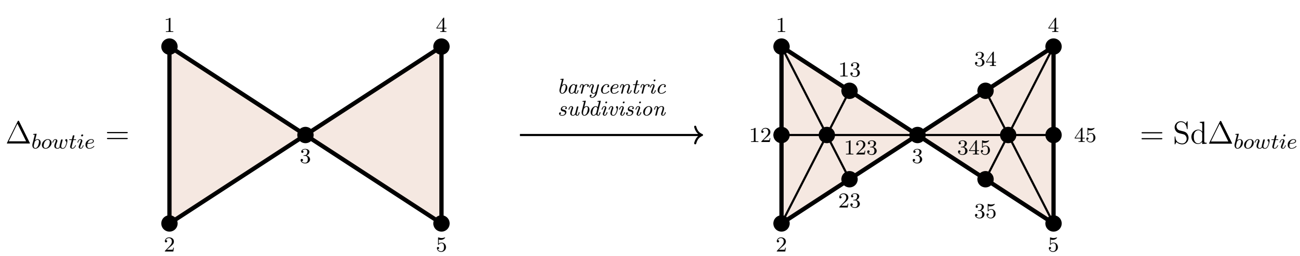
Definition 3.1.5 (A grading specialization). *In order to recover* $\Bbbk[\mathrm{Sd}\Delta]$ *as a ring from the* $\Bbbk$*-vector space* $\Bbbk[\mathrm{Sd}\Delta]$ *one may apply a specialization map*

$$\begin{aligned} \nu : \mathbb{N}^d &\longrightarrow \mathbb{N} \\ \epsilon_j &\longmapsto j \end{aligned}$$

Example 3.1.6. *Let* Δ *be a simplicial complex on* $V_\Delta = \{1, 2, 3, 4\}$ *with facets* $F_1 = \{1, 2, 3\}$ *and* $F_2 = \{3, 4, 5\}$*. Under the barycenteric subdivison map we obtain the complex* $\mathrm{Sd}\Delta$ *on vertex set*

$$V_{\mathrm{Sd}\Delta} = \{1, 2, 3, 4, 12, 13, 23, 34, 35, 45, 123, 345\}$$

given here:



The Stanley-Reisner rings of Δ *and* $\mathrm{Sd}\Delta$ *are the quotients*

$$\Bbbk[\Delta] = \frac{\Bbbk[x_1, x_2, x_3, x_4, x_5]}{\langle x_1x_4, x_1x_5, x_2x_4, x_2x_5\rangle.}$$

and

$$\Bbbk[\mathrm{Sd}\Delta] = \frac{\Bbbk[y_1, y_2, y_3, y_4, y_5, y_{12}, y_{13}, y_{23}, y_{34}, y_{35}, y_{45}, y_{123}, y_{345}]}{\langle y_1y_2, y_1y_3, y_1y_4, y_1y_5, y_2y_3, y_2y_4, y_2y_5, y_3y_4, y_3y_5, y_4y_5, \cdots y_{34}y_{123}, y_{35}y_{123}, y_{45}y_{123}\rangle}.$$

Without even writing out the entire ideal for $\Bbbk[\mathrm{Sd}\Delta]$, *one can already see that* $I_{\mathrm{Sd}\Delta}$ *is* much *larger than that of* $\Bbbk[\Delta]$.

3.1.1. Parameter Rings for $\Bbbk[\Delta]$ and $\Bbbk[\mathrm{Sd}\Delta]$. Let Δ be a $d-1$-dimensional simplicial complex and let $A := \Bbbk[z_1, \dots, z_d]$. Recall from section 2.4 the universal system of parameters $\theta = (\theta_1, \dots, \theta_d)$ for a Stanley-Reisner ring, $\Bbbk[\Delta]$, is a homogeneous system of parameters. Thus, $\Bbbk[\Delta]$ is a finitely generated $\mathbb{N}$-graded A-module under the ring map

$$\begin{aligned} A &\longrightarrow \Bbbk[\Delta] \\ z_j &\longmapsto \theta_j \text{ for } j = 1, 2, \dots, d. \end{aligned}$$

Similarly, for the barycentric subdivision $\mathrm{Sd}\Delta$ of Δ, the colorful system of parameters, $\gamma = (\gamma_1, \dots, \gamma_d)$, is a homogeneous system of parameters to $\Bbbk[\mathrm{Sd}\Delta]$. Thus, $\Bbbk[\mathrm{Sd}\Delta]$ is finitely generated as a $\mathbb{N}^d$-graded A-module under the ring map

$$\begin{aligned} A &\longrightarrow \Bbbk[\mathrm{Sd}\Delta] \\ z_j &\longmapsto \gamma_j \text{ for } j = 1, 2, \dots, d. \end{aligned}$$

Remark 3.1.7. *From here on, we will consider* $\Bbbk[\mathrm{Sd}\Delta]$ *to be* $\mathbb{N}$*-graded under the grading specialization in definition 3.1.5.*

Example 3.1.8. *Using the same simplicial complex* Δ *and* $\mathrm{Sd}\Delta$ *from Example 3.1.6 The universal system of parameters in* $\Bbbk[\Delta]$ *is*

$$\theta = (x_1 + x_2 + x_3 + x_4 + x_5, x_1x_2 + x_1x_3 + x_2x_3 + x_3x_4 + x_3x_5 + x_4x_5, x_{123} + x_{345}).$$

Given that $\deg_{\mathbb{N}}(\theta_i) = i$ *we may write* $\Bbbk[\Delta]$ *as an* $\mathbb{N}$*-graded* $\Bbbk[\theta]$*-module. On the other side of things we have the colorful system of parameters*

$$\gamma = (y_1 + y_2 + y_3 + y_4 + y_5, y_{12} + y_{13} + y_{23} + y_{34} + y_{35} + y_{45}, y_{123} + y_{345})$$

inside $\Bbbk[\mathrm{Sd}\Delta]$. *The* $\mathbb{N}^3$ *grading is* $\deg_{\mathbb{N}^3}\gamma_i = \varepsilon_i$. *Thus,* $\Bbbk[\mathrm{Sd}\Delta]$ *as a* $\mathbb{N}^3$*-graded* $\Bbbk[\gamma]$*-module.*

REMARK 3.1.9. *Since there is an obvious map between* $\Bbbk[\Delta]$ *and* $\Bbbk[\mathrm{Sd}\Delta]$ *sending* $\theta_i \mapsto \gamma_i$, $1 \le i \le d$, *by mapping* $\mathbf{x}^F \mapsto y_F$. *This map fails to be a ring isomorphism and instead lingers at merely a* $\Bbbk$-*vector space isomorphism.*

3.2. A colorful Hochster formula

As discussed in Section 2.3, Hochster's formula, stated in Theorem 2.3.5, is useful for computing the graded Betti numbers. This formula was generalized in [1] where a "colorful" version is given. We use this section to give this generalization.

DEFINITION 3.2.1. *Let* Δ *be a simplicial complex on a vertex set* V. *A map* $\kappa : V \to [d]$ *is an* (proper, vertex-) d-coloring *of* Δ *if for any edge* $E \in \Delta$ *on vertices* $i, j \in V$, $\kappa(i) \neq \kappa(j)$. *In particular, when* $d = \dim(\Delta) + 1$ *the coloring is called* balanced.

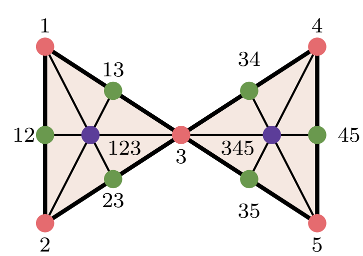
A proper, vertex d-coloring κ is only dependent on the 1-skeleton of a simplicial complex. It also allows us a natural setting for defining an $\mathbb{N}^d$-multigrading on $\Bbbk[\Delta]$. If $x_i \in \Bbbk[\Delta] = S/I_\Delta$, then $\deg(x_i) := \varepsilon_{\kappa(i)}$ where ε_j is the j-th standard basis vector in $\mathbb{N}^d$. It is easy to check that grading preserves homogeneity of I_Δ.

DEFINITION 3.2.2 (Colorful SOP i). *Let* Δ *be a simplicial complex with a* d*-coloring* κ *and let* $\gamma = (\gamma_1, \dots, \gamma_d)$ *where*

$$\gamma_j = \sum_{i \in \kappa^{-1}(j)} y_i.$$

REMARK 3.2.3. *When the simplicial complex is balanced, then the colorful SOP are a homogeneous system of parameters.*

EXAMPLE 3.2.4. *We return to the example from above, representing the coloring with actual colors.*

$\mathrm{Sd}\Delta_{bowtie} =$ 

The colorful system of parameters for $\Bbbk[\mathrm{Sd}\Delta]$ *are*

$$\gamma_1 = y_1 + y_2 + y_3 + y_4 + y_5$$

$$\gamma_2 = y_{12} + y_{13} + y_{23} + y_{34} + y_{35} + y_{45}$$

$$\gamma_3 = y_{123} + y_{345}.$$

This gives us the parameter ring $\Bbbk[\gamma_1, \gamma_2, \gamma_3]$.

In [1], it is shown that $\Bbbk[\Delta]$ is finitely-generated as an $\mathbb{N}^d$-graded module over the polynomial ring $A := \Bbbk[z_1, \ldots, z_d]$ via the ring map $A \to \Bbbk[\Delta]$, $z_j \mapsto \gamma_j$. It is natural to ask about the structure of $\Bbbk[\Delta]$ as an A-module, thus we first ask for the shape of the (minimal, finite-free) resolution of $\Bbbk[\Delta]$ as an A-module. The shape of this resolution is given by the $\mathbb{N}^d$-graded Betti numbers and is determined by the isomorphism in the following theorem.

Theorem 3.2.5. **(Colorful Hochster formula, [1])**
Let Δ *be a simplicial complex with fixed coloring* κ. *If* $\Bbbk[\Delta]$ *is the Stanley-Reisner ring of* Δ *and* $\gamma = (\gamma_1, \ldots, \gamma_d)$ *is a colorful system of parameters, then if we view* $\Bbbk[\Delta]$ *as an* $\mathbb{N}^d$*-graded* A*-module, where ring map* $A = \Bbbk[z_1, \ldots, z_d] \to \Bbbk[\Delta]$, $z_j \mapsto \gamma_j$, *we have*

$$\mathrm{Tor}_i^A(\Bbbk[\Delta], \Bbbk)_{\mathbf{a}} \cong \begin{cases} 0 & \text{if } \mathbf{a} \notin \{0,1\}^d, \\ \tilde{H}^{\#S-i-1}(\Delta|_S, \Bbbk) & \text{if } \mathbf{a} = \sum_{j \in S} \epsilon_j \in \{0,1\}^d, \end{cases}$$

for any $\mathbf{a} \in \mathbb{N}^d$.

Furthermore, these $\Bbbk$-vector space isomorphisms are equivariant with respect to the automorphism group of Δ with respect to the coloring κ. When a simplicial complex is given a trivial coloring, i.e., every vertex is a different color, then the above formula reduces to Theorem 2.3.5. However, our purposes require a balanced d-coloring since our primary use of the colorful Hochster formula is to predict the shape of the free resolution of $\Bbbk[\mathrm{Sd}\Delta]$ over A.

3.3. Conjectures on Stanley-Reisner rings

One may have observed from Example 3.2.4 that there is another way to define the colorful system of parameters when the simplicial complex is the Barycentric subdivision of a simplicial complex.

PROPOSITION 3.3.1 (Colorful SOP ii). *Let* Δ *be a simplicial complex with Barycentric subdivision* $\mathrm{Sd}\Delta$ *then if* $\gamma = (\gamma_1, \ldots, \gamma_d)$ *is a Colorful SOP, then*

$$\gamma_j = \sum_{\substack{F \in \Delta \\ |F| = j}} y_F.$$

PROOF. We need only show that the two definitions of γ are equivalent. Label $\mathrm{Sd}\Delta$ so that if v is a vertex of $\mathrm{Sd}\Delta$, then v is the Barycenter of a face $F \in \Delta$. Set $v = F$. Then $\Bbbk[\mathrm{Sd}\Delta]$ has the presentation as in Definition 3.1.1. Since $\mathrm{Sd}\Delta$ is a Barycentric subdivison, then it is possible to give $\mathrm{Sd}\Delta$ a balanced d-coloring, i.e., $d = \dim(\mathrm{Sd}\Delta) + 1 = \dim(\Delta) + 1$. Let κ denote the coloring so that for $v \in V$, $\kappa(v) = \dim_\Delta(v) + 1$ where $\dim_\Delta$ denotes the dimension of $v \in \Delta$. This is a proper, balanced d-coloring which induces an $\mathbb{N}^d$-grading. The result then follows naturally. □

Observe that we can view both $\Bbbk[\Delta]$ and $\Bbbk[\mathrm{Sd}\Delta]$ as A-modules. Our goal is to compare $\Bbbk[\Delta]$ and $\Bbbk[\mathrm{Sd}\Delta]$ as A-modules were z_i acts by multiplication by θ_i on $\Bbbk[\Delta]$ and z_i acts by multiplication by γ_i on $\mathrm{Sd}\Delta$. We state two conjectures made in [1]. In Conjecture 3.3.2, we ask if the resolutions of $\Bbbk[\mathrm{Sd}\Delta]$ has the same resolution shape as $\Bbbk[\Delta]$ under the grading specialization from Definition 3.1.5 when both are regarded as $\mathbb{N}$-graded A-modules, i.e., do they have the same graded Betti numbers. In Conjecture 3.3.4, we ask if the first two maps in the resolutions are equal, i.e., are $\Bbbk[\Delta]$ and $\Bbbk[\mathrm{Sd}\Delta]$ isomorphic as $\mathbb{N}$-graded A-modules.

CONJECTURE 3.3.2. *For any simplicial complex* Δ *of dimension* $d-1$ *with barycentric subdivision* $\mathrm{Sd}\Delta$*, for each* $i = 0, 1, \ldots, d$*,*

$$\mathrm{Tor}_i^A(\Bbbk[\Delta], \Bbbk)_j \cong \mathrm{Tor}_i^A(\Bbbk[\mathrm{Sd}\Delta], \Bbbk)_j \cong \bigoplus_{\substack{S \subseteq [d] \\ j = \sum_{s \in S}}} \widetilde{H}^{|S|-i-1}((\mathrm{Sd}\Delta)|_S, \Bbbk) \tag{3.1}$$

as vector spaces.

REMARK 3.3.3. *The original motivation was to understand the action of a subgroup* $\mathrm{Aut}_\kappa(\mathrm{Sd}\Delta)$ *of the group of simplicial automorphisms* $\mathrm{Aut}(\mathrm{Sd}\Delta)$*,*

$$\mathrm{Aut}_\kappa(\mathrm{Sd}\Delta) := \{g \in \mathrm{Aut}(\Delta) \mid \kappa(g.i) = \kappa(i) \textit{ for all } i \textit{ in } [n]\}.$$

This group contains those $\mathbb{N}^d$*-graded simplicial automorphims which preserve the balanced, proper coloring of* $\mathrm{Sd}\Delta$*. Since* $\Bbbk[\mathrm{Sd}\Delta]$ *is* $\mathbb{N}^d$*-graded, i.e., graded by color, then* $\Bbbk[\mathrm{Sd}\Delta] = \bigoplus_{\mathbf{a} \in \mathbb{N}^d} \Bbbk[\mathrm{Sd}\Delta]_{\mathbf{a}}$, *where* $\Bbbk[\mathrm{Sd}\Delta]_{\mathbf{a}}$ *is the* $\mathbf{a}$*-th graded piece. Not only is* $\Bbbk[\mathrm{Sd}\Delta]_{\mathbf{a}}$ *a* $\Bbbk$*-vector space, it is a representation of* $\mathrm{Aut}_\kappa \Bbbk(\mathrm{Sd}\Delta)$*, which means it's a module over the group algebra* $\Bbbk[\mathrm{Aut}_\kappa(\mathrm{Sd}\Delta)]$*. For a subgroup* $G \subseteq \mathrm{Aut}_\kappa(\mathrm{Sd}\Delta)$*, the Grothendieck ring* $R_\Bbbk(G)$ *is used to keep track of these representations. For any class* $[U]$ *of* $R_\Bbbk(G)$*, the dimension homomorphism* $R_\Bbbk(G) \to \mathbb{Z}$*,* $[U] \mapsto \dim_\Bbbk U$ *allows us to ignore the* $\Bbbk G$*-module structure and phrase equivariant statements in terms of non-equivariant statements. We now give the original conjecture as stated in [1] specialized to the case of* Δ *being only a simplicial complex:*

For any simplicial complex Δ *of dimension* $d-1$*, and any subgroup* G *of* $\mathrm{Aut}(\Delta)$*, for each* $m = 0, 1, \ldots, d$ *one has these equalities in* $R_\Bbbk(G)$*:*

$$\left[\mathrm{Tor}_m^{\Bbbk[\Theta]}(\Bbbk[\Delta], \Bbbk)_j\right] = \left[\mathrm{Tor}_m^{\Bbbk[\Gamma]}(\Bbbk[\mathrm{Sd}\Delta], \Bbbk)_j\right] = \sum_{\substack{S \subseteq [d]: \\ j = \sum_{s \in S} s}} \left[\tilde{H}^{\#S-m-1}\left((\mathrm{Sd}\Delta)\,|_S, \Bbbk\right)\right]$$

Equivalently, one has this equality in $R_\Bbbk(G)[[t]]$*:*

$$\mathrm{Hilb}_{\mathrm{eq}}(\mathrm{Tor}_m^{\Bbbk[\Theta]}(\Bbbk[\Delta], \Bbbk), t) = \left[\mathrm{Hilb}_{\mathrm{eq}}(\ \mathrm{Tor}_m^{\Bbbk[\Gamma]}(\Bbbk[\mathrm{Sd}\Delta], \Bbbk),\ t_1, \ldots, t_d)\right]_{\substack{t_1 = t \\ t_2 = t^2 \\ \vdots \\ t_d = t^d}} . \tag{3.2}$$

When $\Bbbk G$ *is semisimple, the first line of equalities in the conjecture would be isomorphisms:*

$$\mathrm{Tor}_m^{\Bbbk[\Theta]}(\Bbbk[\Delta], \Bbbk)_j \cong \mathrm{Tor}_m^{\Bbbk[\Gamma]}(\Bbbk[\mathrm{Sd}\Delta], \Bbbk)_j = \bigoplus_{\substack{S \subseteq [d]: \\ j = \sum_{s \in S} s}} \tilde{H}^{\#S-m-1}\left((\mathrm{Sd}\Delta)\,|_S, \Bbbk\right)$$

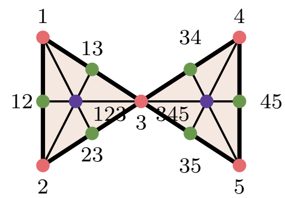
This happens, e.g., if one ignores the group action by taking $G = \{1\}$*, or more generally, when* $\#G \in \Bbbk^\times$*.*

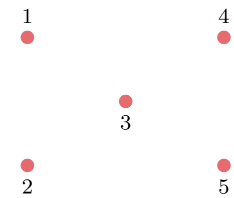
Conjecture 3.3.4. *[1] Since the universal system of parameters and the colorful system of parameters can be seen to generate the same* $\mathbb{N}$*-graded* $A = \Bbbk[z_1, \ldots, z_d]$ *subalgebra,* $z_i := \sum_{\substack{F \in \Delta: \\ \dim_\Delta(F) = i}} y_F$*, we may write* $\Bbbk[\Delta]$ *and* $\Bbbk[\mathrm{Sd}\Delta]$ *as* $\mathbb{N}$*-graded* A*-modules. When viewed as such, there exists an equivariant isomorphism between* $\Bbbk[\Delta]$ *and* $\Bbbk[\mathrm{Sd}\Delta]$*.*

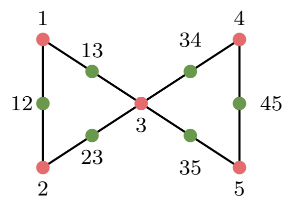
Note that in [1], Conjecture 3.3.4 is written as only a question. However, since the paper has been uploaded to the `arXiv`, the question has been upgraded to a conjecture due to more solid evidence that it holds.

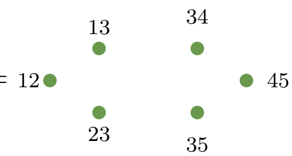
Remark 3.3.5. *The conjectures in [1] are much more general than we are stating here. In the paper, we allow* Δ *to be a finite- regular- CW-complex whose maximal cells look like simplicies (i.e., they come from a structure called a simplicial poset) and the ring* $\Bbbk[\Delta]$ *is a generalized version of the Stanley-Reisner ring called the face ring of an object that Richard Stanley calls simplicial posets, i.e., the face poset of* Δ*. For more specifics on the generalized versions, we direct the reader to the original motivating paper [1] and Stanley's book [15]. Ultimately, it would be wonderful to show these conjectures hold for CW-complexes, however at this time we focus on the case when* Δ *is only a simplicial complex.*

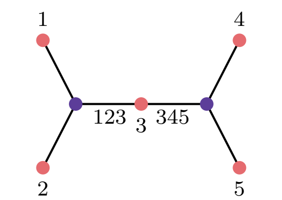
Example 3.3.6. *We give an example of Conjecture 3.3.2. First, recall that under the grading specialization* $\mathrm{Sd}\Delta$ *is* $\mathbb{N}$*-graded by colors* $\{1,2,3\}$ *where* $\deg(y_F) = \dim_\Delta(F) + 1$*. We continue with the simplicial complex from Example 3.1.6 and Example 3.1.8. First, we begin by calculating the color restricted subcomplexes of* $\mathrm{Sd}\Delta$.

$(\mathrm{Sd}\Delta|)_{\{1,2,3\}} =$ 

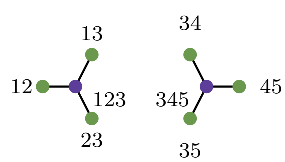
$(\mathrm{Sd}\Delta|)_{\{1\}} =$ 

$(\mathrm{Sd}\Delta|)_{\{1,2\}} =$ 

$(\mathrm{Sd}\Delta|)_{\{2\}} =$ 

$(\mathrm{Sd}\Delta|)_{\{1,3\}} =$ 

$(\mathrm{Sd}\Delta|)_{\{3\}} =$ 

$(\mathrm{Sd}\Delta|)_{\{2,3\}} =$ 

$(\mathrm{Sd}\Delta|)_{\varnothing} = \varnothing$

We next calculate the Tor *groups given the colorful Hochster formula from 3.2.5. For calculating* $\mathrm{Tor}_0^A(\Bbbk[\Delta], \Bbbk)_j$ *we have the following non-zero reduced cohomology groups:*

j	$S \subseteq [4]$	*Reduced Cohomology*	*Resolution Factor*
0	$\varnothing$	$\widetilde{H}^{0-0-1}((\mathrm{Sd}\Delta)\vert_{\varnothing}, \Bbbk) = \Bbbk^1$	$A(-0)^1$
1	$\{1\}$	$\widetilde{H}^{1-0-1}((\mathrm{Sd}\Delta)\vert_{\{1\}}, \Bbbk) = \Bbbk^4$	$A(-1)^4$
2	$\{2\}$	$\widetilde{H}^{1-0-1}((\mathrm{Sd}\Delta)\vert_{\{2\}}, \Bbbk) = \Bbbk^5$	$A(-2)^5$
3	$\{3\}, \{1,2\}$	$\widetilde{H}^{1-0-1}((\mathrm{Sd}\Delta)\vert_{\{3\}}, \Bbbk) \oplus \widetilde{H}^{2-0-1}((\mathrm{Sd}\Delta)\vert_{\{1,2\}}, \Bbbk) = \Bbbk^3$	$A(-3)^4$.

For calculating $\mathrm{Tor}_1^A(\Bbbk[\Delta],\Bbbk)_j$ *we have the following non-zero reduced cohomology groups:*

j	$S\subseteq[4]$	*Reduced Homology*	*Resolution Factor*
5	$\{2,3\}$	$\widetilde{H}^{2-1-1}((\mathrm{Sd}\Delta)\vert_{\{2,3\}},\Bbbk)=\Bbbk^1$	$A(-5)^1$.

One may check that the rest are zero. Conjecture 3.3.2 tells us that by colorful Hochster formula, the resolution of $\Bbbk[\Delta]$ *as an A-module is*

$$\mathcal{F}_\bullet^\Delta:\quad 0\longleftarrow \Bbbk[\Delta]\xleftarrow{\partial_0} \begin{matrix}A(-0)^1\\ \oplus\\ A(-1)^4\\ \oplus\\ A(-2)^5\\ \oplus\\ A(-3)^3\end{matrix}\longleftarrow A(-5)^1\longleftarrow 0.$$

One may verify using the package `ResolutionsOfStanleyReisnerRings` *in* `Macaulay2`*.*

CHAPTER 4

The Coinvariant algebra and its Gröbner basis

4.1. Monomial Orderings

A Gröbner basis, which will be discussed in the next section, is dependent on a monomial ordering. In some contexts, this is also called a term order. However, we are then faced with a problem: When given two monomials, how do we say that one is larger than the other? In this section we will define monomial orderings and give three common monomial orderings.

Definition 4.1.1 (A Monomial Ordering). *Let $S = \Bbbk[x_1, \ldots, x_n]$ be a polynomial ring. A* monomial ordering *is a relation $>$ on the set of monomials $\mathbf{x}^{\mathbf{a}}$, $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}$ that satisfies:*

(1) $>$ is a total (or linear) order on $\mathbb{Z}^n_{\geq 0}$;

(2) If $\mathbf{x}^{\mathbf{a}} > \mathbf{x}^{\mathbf{b}}$, then $\mathbf{x}^{\mathbf{a}+\mathbf{c}} > \mathbf{x}^{\mathbf{b}+\mathbf{c}}$, $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{Z}^n_{\geq 0}$;

(3) $>$ is a well-ordering on $\mathbb{Z}^n_{\geq 0}$.

Definition 4.1.2. *We give definitions and examples here. Recall that if $\mathbf{x}^{\mathbf{a}} \in S = \Bbbk[x_1, \ldots, x_n]$, then* $\deg(\mathbf{x}^{\mathbf{a}}) = a_1 + \cdots + a_n$.

Name	Definition: $\mathbf{x}^{\mathbf{a}} < \mathbf{x}^{\mathbf{b}}$ iff...	Example
Lexicographic (lex)	*there exists* $1 \leq i \leq n$ *such that* $a_1 = b_1, a_2 = b_2, \ldots, a_{i-1} = b_{i-1}$, *and* $a_i < b_i$.	$x_1^2 > x_1x_2 > x_1 > x_2^2$
Degree reverse lexicographic (degrevlex)	$\deg(\mathbf{x}^{\mathbf{a}}) < \deg(\mathbf{x}^{\mathbf{b}})$ *or* $\deg(\mathbf{x}^{\mathbf{a}}) = \deg(\mathbf{x}^{\mathbf{b}})$ *and there exists* $1 \leq i \leq n$ *such that* $a_1 = b_1, a_2 = b_2, \ldots, a_{i-1} = b_{i-1}$, *and* $a_i > b_i$.	$x_1^2 > x_1x_2 > x_1x_3 > x_2^2$
Degree lexicographic (deglex)	$\deg(\mathbf{x}^{\mathbf{a}}) < \deg(\mathbf{x}^{\mathbf{b}})$ *or* $\deg(\mathbf{x}^{\mathbf{a}}) = \deg(\mathbf{x}^{\mathbf{b}})$ *and there exists* $1 \leq i \leq n$ *such that* $a_1 = b_1, a_2 = b_2, \ldots, a_{i-1} = b_{i-1}$, *and* $a_i < b_i$.	$x_1^2 > x_1x_2 > x_2^2 > x_1x_3$

4.2. Gröbner basis

A Gröbner basis is a way of transforming a set of polynomials algorithmically into a generating set for an ideal. [16] Let $R = \Bbbk[x_1, \ldots, x_n]$ be a multivariate polynomial ring over a field $\Bbbk$. Given an monomial ordering, $\mathcal{G}$ is a Gröbner basis of an ideal I of R if the leading term ideal $\mathrm{LT}(\mathcal{G})$ is equal to the leading term ideal $\mathrm{LT}(H)$, where H is the set generating I. (In some literature, the leading term ideal is referred to as the "initial ideal.") For our purposes, we will be using degrevlex (Definition 4.1.2).

The algorithm often used to compute this basis is called the Buchberger algorithm. In this section, we will define Gröbner bases and give some of the necessary machinery that is necessary for running the Buchberger algorithm.

Let $\mathrm{LCM}(f, g)$ be the least common multiple for polynomials f, g, and define $\mathrm{LT}(f)$ to be the leading term of a polynomial f. The most important tool for the Buchberger algorithm will be the S-polynomial, where S stands for "syzygy."

Definition 4.2.1 (S-polynomial). *Let* $f, g \in S$ *be polynomials. The* S-polynomial *is*

$$S(f, g) := \frac{\mathrm{LCM}(\mathrm{LM}(f), \mathrm{LM}(g))}{\mathrm{LT}(f)} \cdot f - \frac{\mathrm{LCM}(\mathrm{LM}(f), \mathrm{LM}(g))}{\mathrm{LT}(g)} \cdot g.$$

The pair f, g are called a critical pair.

If both p and q belong to the same ideal $I \subset R$, then $S(p, q) \in I$, thereby dealing with the ideal membership problem.

THEOREM 4.2.2 (Buchberger Criterion). *$\mathcal{G}$ is a Gröbner basis if and only if for all pairs $p, q \in \mathcal{G}$ $S(f, g) \equiv 0$ modulo $\mathcal{G}$.*

This is the necessary condition to write down the algorithm that allows us to obtain a Gröbner basis of an ideal I.

ALGORITHM 4.2.3 (Buchberger algorithm). ***Input** A set of polynomials A. **Output** A Gröbner basis $\mathcal{G}$.*

(1) $\mathcal{G} := F$.

(2) For all $g, f \in \mathcal{G}$, reduce the S-polynomial $S(f, g)$ until $S(f, g)$ is irreducible with respect to $\mathcal{G}$ via the multivariate division algorithm.

(3) If the remainder of $S(f, g)$ after division by the elements in $\mathcal{G}$ (this is the result of step (2)) is nonzero, then add it to the set $\mathcal{G}$.

(4) Continue this process until $S(p, q) = 0$ with respect to $\mathcal{G}$ for all $p, g \in \mathcal{G}$. Note that this will also include those polynomials added via step (3).

An important use of Gröbner basis comes from the power of the following theorem.

THEOREM 4.2.4. *Let R be a commutative ring in n variables, and let $I \subset R$ be a homogeneous ideal of S. Then the leading term ideal $\mathrm{LT}(\mathcal{G}) = \{g_1, \ldots, g_k\}$ of the Gröbner basis $\mathcal{G}$, is also homogeneous,*

$$\mathrm{Hilb}_R(R/I; q) = \mathrm{Hilb}_R(R/\mathrm{LT}(\mathcal{G}); q)$$

When the terms in $\mathrm{LT}(\mathcal{G})$ form a regular sequence we have

$$\mathrm{Hilb}_R(R/\mathrm{LT}(\mathcal{G}); q) = \frac{(1 - q^{\deg(g_1)}) \cdots (1 - q^{\deg(g_k)})}{(1 - q)^n}.$$

PROOF. The proof that the Hilbert series are equal can be found in [6, Thm. 15.26]. □

REMARK 4.2.5. *Note that the leading terms to not necessarily form a regular sequence. For example, take the leading term ideal $\mathrm{LT}(\mathcal{G}) = (x^5, y^5, x^4y^4)$ which is both minimal and reduced inside the ideal they generate. However, they do not form a regular sequence.*

Computationally, this means we may understand properties of R/I by studying the initial ideal $\mathrm{LT}(\mathcal{G}))$ of the Gröbner basis of the ideal I. Since $\mathrm{LT}(\mathcal{G})$ is generated by monomials, this ideal is often easier to understand than the original ideal I. In particular, it is quite useful in the case when one has an ideal $I = J_1 + J_2$ composed of two ideals $J_1, J_2 \subseteq R$. By computing the Gröbner basis of I, this larger ideal is now relatively tractable. This being said, as we have seen, there are complexity issues with computing Gröbner bases and they are often better used in actual computation by using math computational software such as `Macaulay2` or `SageMath`.

4.3. The Coinvariant Algebra

4.3.1. Symmetric polynomials. Recall from Chapter 2 our discussion on the elementary symmetric functions. In this section we will use another set of symmetric functions known as the homogeneous symmetric functions. We give both of their definitions here:

- the *elementary symmetric functions*

$$e_k(\mathbf{x}_1) = \sum_{1 \leq j_1 \leq \cdots \leq j_k \leq n} x_{j_1} \cdots x_{j_k}$$

  where $\mathbf{x}_1 = (x_1, x_2, \ldots, x_n)$ is a list of variables and

- the *homogeneous symmetric functions*

$$h_k[\mathbf{x}_t] = \sum_{k \leq j_1 \leq \cdots \leq j_k \leq n} x_{j_1} \cdots x_{j_k}$$

  where $\mathbf{x}_t = (x_t, x_{t+1}, \ldots, x_n)$ is also a list of variables.

For the following work it is important to remark that the input variables to both of these these functions do not necessarily include $x_1, \ldots, x_n$ and are allowed to vary. When we drop the subscript so that $\mathbf{x}_1 = \mathbf{x}$ it means that it is the entire list of variables $x_1, \ldots, x_n$. The following is a well known proposition.

Proposition 4.3.1. *Let $\Bbbk[x_1, \ldots, x_n]$ be a polynomial ring. The reduced Gröbner basis for the ideal generated by the elementary symmetric functions, $e_1(\mathbf{x}), \ldots, e(\mathbf{x})$, is given by*

$$h_1[\mathbf{x}_1], h_2[\mathbf{x}_2], \ldots, h_n[\mathbf{x}_n].$$

EXAMPLE 4.3.2. *If we consider $n = 3$ the elementary symmetric functions are*

$$e_1(x_1, x_2, x_3) = x_1 + x_2 + x_3 \quad e_2(x_1, x_2, x_3) = x_1x_2 + x_1x_3 + x_2x_3 \quad e_3(x_1, x_2, x_3) = x_1x_2x_3$$

and the homogeneous symmetric functions which form a Gröbner basis for the ideal $(e_1(\mathbf{x}), e_2(\mathbf{x}), e_3(\mathbf{x}))$ *are*

$$h_1(x_1, x_2, x_3) = x_1 + x_2 + x_3 \quad h_2(x_2, x_3) = x_2^2 + x_2x_3 + x_3^2 \quad h_3(x_3) = x_3^3.$$

4.3.2. A Gröbner basis for the coinvariant algebra. Let $S = \Bbbk[x_1, \ldots, x_n]$ be a polynomial ring over a field $\Bbbk$ of characteristic 0. The symmetric group S_n acts on S by permuting the variables, giving us the S^n-invariant subring,

$$S^{\mathfrak{S}_n} = \{f \in S \mid \sigma.f = f \text{ for all } \sigma \in \mathfrak{S}_n\}.$$

This algebra is generated by the elementary symmetric functions given in variables $x_1, \ldots, x_n$. So, the ideal $(S_+^{\mathfrak{S}_n}) = (e_1(\mathbf{x}), \ldots, e_n(\mathbf{x}_n)) \subset S$ is invariant under $\mathfrak{S}_n$-action. By considering the quotient of S by this ideal, we obtain the *coinvariant algebra,* $\mathcal{A}_n = \Bbbk[x_1, \ldots, x_n]/(e_1(\mathbf{x}), \ldots, e_n(\mathbf{x}))$. This is a well studied algebra. For more on this topic we direct the reader towards [8, 9].
In Proposition 4.3.1 we stated that a reduced Gröbner basis for $I_n = (e_1(\mathbf{x}), \ldots, e_n(\mathbf{x}))$ is given by $h_i[\mathbf{x}_i]$ for all $1 \leq i \leq n$. Then the basis for $\mathcal{A}_n$ is given by the set $\{x_1^{a_1} \cdots x_n^{a_n} \mid 0 \leq a_i \leq i-1 \text{ for all } i\}$. These are called the *substaircase monomials* but are also known as the *Artin basis.* [2]. From here it is easy to see that $\mathcal{A}$ has vector space dimension $n!$. For more on this particular decomposition we refer the reader to Bergeron's book [3, Section 7.2] and Sturmfels' book [17, Theorem 1.2.7].

THEOREM 4.3.3. *If $S = \Bbbk[x_1, \ldots, x_n]$ and let $I_n = (e_1(\mathbf{x}), \ldots, e_n(\mathbf{x}))$, then $\mathcal{A}_n = S/I_n$ is the coinvariant algebra. Then*

$$\text{Hilb}_S(A_n; q) = [n]_q!$$

where

$$[n]_q! := [n]_q[n-1]_q \cdots [1]_q \quad \textit{and} \quad [n]_q := 1 + q + q^2 + \cdots + q^n.$$

Proof. The coinvariant algebra $\mathcal{A}_n$ has the Gröbner basis

$$\mathcal{G} = \{h_1[\mathbf{x}_1], h_2[\mathbf{x}_2], \ldots, h_n[\mathbf{x}_n]\}.$$

The leading term ideal is then

$$\mathrm{LT}(\mathcal{G}) = \{x_1, x_2^2, \ldots, x_n^n\}.$$

The elements of the leading term ideal of a Gröbner basis form a regular sequence in $\mathcal{A}$, so by Theorem 4.2.4,

$$\mathrm{Hilb}_S(\mathcal{A}_n; q) = \mathrm{Hilb}_S(S/\mathrm{LT}(\mathcal{G}; q) = \frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q)^n}$$

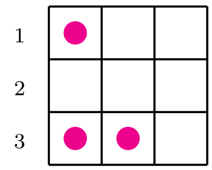
By multiplying out the above computation, we obtain the desired result. □

4.3.3. A combinatorial method for computing the Artin basis. We now give a combinatorial method for computing the Artin basis. First, we define a combinatorial object for computing monomials. Let $S = \Bbbk[x_1, \ldots, x_n]$. Draw an $n \times n$ box and label the rows on the left by $1, \ldots, n$ beginning at the northwest corner. These represent the possible choices $1, \ldots, n$ for support of a monomial $x_1^{a_1} \cdots x_n^{a_n}$. To determine $a_1, \ldots, a_n$, we follow the following rules:

(1) Each box can hold a maximum of 1 ball.
(2) Balls can only be placed in boxes which are not filled. We call unfilled boxes *open*.
(3) Balls must be left-aligned, i.e., we begin on the west side and find the next open box in whichever row we are attempting to place a ball.
(4) For all $1 \leq i \leq n$, each ball in row i contributes to the power of the variable x_i in the monomial $x_1^{a_1} \cdots x_n^{a_n}$, i.e., for the variable $x_i^{a_i}$,

$$a_i = \#\{\text{balls in row } i\}$$

Example 4.3.4. *Let $n = 3$ so that we have monomials $\mathbf{x}^{\mathbf{a}} \in S = \Bbbk[x_1, x_2, x_3]$. Then we have the corresponding monomials and boxes:*



$x_1x_3^2$

To obtain the Artin basis, we draw a path along the diagonal beginning in the northwest corner and proceeding south one box, then walking east one box, then walking south one box, etc.. We then fill in everything above the diagonal. The resulting box is given in Figure 4.1. Since everything above the diagonal is filled, then the only boxes in which we may place a ball lie below the diagonal.

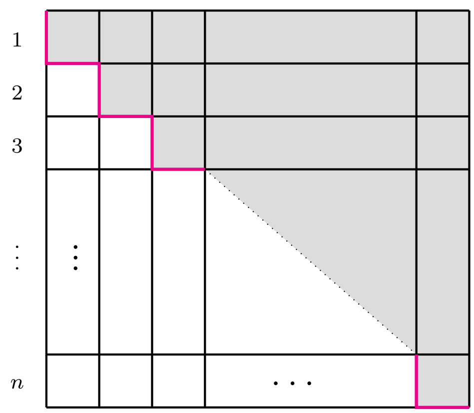


FIGURE 4.1. A combinatorial method for computing the Artin basis for $\mathcal{A}_n$

PROPOSITION 4.3.5. *The possible fillings of an $n \times n$-box restricted to below the diagonal as described above are in bijection with the substaircase monomials in n variables.*

PROOF. Recall that the set of substaircase monomials are given by

$$\{x_1^{a_1} \cdots x_n^{a_n} \mid 0 \leq a_i \leq i-1 \text{ for all } i\}.$$

By inducting on i we can see that for each row $1 \leq j \leq i$

$$\#\{\text{open boxes in row } j\} = a_j$$

so that monomials from the placement of the balls are precicely those monomials in the set

$$\{x_1^{a_1} \cdots x_j^{a_j} \mid 0 \leq a_k \leq j-1 \text{ for all } 1 \leq k \leq j\}.$$

□

Corollary 4.3.6. *Let $S = \Bbbk[x_1, \ldots, x_n]$ and let $I_n = (e_1(\mathbf{x}), \ldots, e_n(\mathbf{x}))$ then*

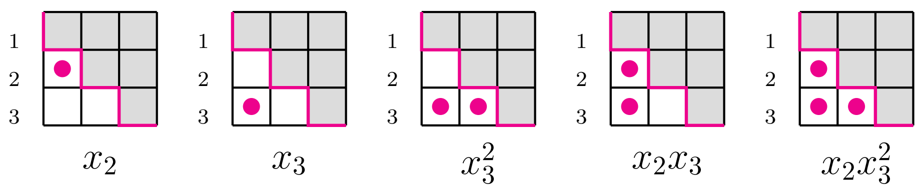
$$\mathrm{Hilb}(S/I_n; q) = \prod_{1 \le i \le n} (1 - q^{\#\{\text{open boxes in row } i\}+1}).$$

Proof.

$$\prod_{1 \le i \le n} (1 - q^{\#\{\text{open boxes in row } i\}+1}) = \prod_{1 \le i \le n} (1 - q^{i+1}) = [n]_q! = \mathrm{Hilb}(S/I_n; q).$$

□

Example 4.3.7. *Let $n = 3$ so that we have monomials $S = \Bbbk[x_1, x_2, x_3]$. To obtain the substaircase monomials for S/I_n we draw the 3×3 box with diagonal marked as below. We then have the following monomials which correspond with possible left-aligned fillings of boxes below the diagonal.*



We may translate the choices for ball placement in the $n \times n$-box into the theory of integer partitions. Let b be the number of total balls placed then $\lambda \vdash b$ is the partition where each part λ_j is equal to the number of balls in row j. We note that these may not be increasing, so we rearrange accordingly, which can be done since our rings are commutative. Let $\mathbf{x}^{\mathbf{a}}$, where $a_i = \#\{\text{balls in row } i\}$, then we say $\mathbf{a}$ is equivalent to a partition λ if $\mathbf{a}$ and λ differ by rearrangement and a monomial is of type λ if its exponent vector is equivalent to λ. This format allows us to count the number of monomials of each type λ whose parts lie under the diagonal.

Lemma 4.3.8. *Let $1 \le b \le n$ and let $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash b$, then the number of ways to uniquely left align $\lambda_1, \ldots, \lambda_k$ under the diagonal inside the box from Figure 4.1 is equal to*

$$\prod_{\substack{1 \le j \le k \\ s.t.\ \lambda_j \ne \lambda_{j-1}}} \binom{n - \lambda_j - \#\{\lambda_i \mid \lambda_i > \lambda_j\}}{\#\{\lambda_i \mid \lambda_i = \lambda_j\}}$$

Proof. Begin by an $n \times n$-box. Let $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash b$. Choose λ_j such that $\lambda_j \neq \lambda_{j-1}$. We now count the number of choices for the left-aligned placement of λ_j in the $n \times n$. By drawing the diagonal and restricting choices for left-aligned placements of the parts of λ, we reduce the number of choices for λ_j to $n - \lambda_j$ positions. However, this is not yet a sufficient restriction. Since we have already placed $\lambda_1, \ldots, \lambda_{j-1}$ and $\lambda_1 \geq \cdots \geq \lambda_{j-1} < \lambda_j$, then we must remove a position for each λ_i for $1 \leq i \leq j-1$. This gives us a total count of $n - \lambda_j - \#\{\lambda_i \mid \lambda_i > \lambda_j\}$ choices of where to place λ_j. Since we are only iterating over distinct λ_j, then from the $n - \lambda_j - \#\{\lambda_i \mid \lambda_i > \lambda_j\}$ choices we must choose $\#\{\lambda_i \mid \lambda_i = \lambda_j\}$ spots to place all λ_i that are equal to λ_j. This gives us the binomial coefficient. By taking the product of each of distinct λ_j in λ, we obtain the final count. □

Theorem 4.3.9. *Let $\mathcal{A}_n = \bigoplus_{b \in \mathbb{N}} (\mathcal{A}_n)_b$ be the $\mathbb{N}$-graded coinvariant ring. Then, for $1 \leq b \leq n$ the dimension of the b-th graded piece*

$$\dim_k (\mathcal{A}_n)_b = \sum_{\lambda \vdash b} \left(\prod_{\substack{1 \leq j \leq \#\lambda \\ s.t.\ \lambda_j \neq \lambda_{j+1}}} \binom{n - \lambda_j - \#\{\lambda_i \mid \lambda_i > \lambda_j\}}{\#\{\lambda_i \mid \lambda_i = \lambda_j\}} \right)$$

Proof. We have shown that the number of ways to place b balls in b left-aligned boxes under the diagonal in an $n \times n$ as in Figure 4.1 is in bijection with the substaircase monomials of degree b. A partition $\lambda \vdash b$ gives us the number of ways to fill $\#\lambda$-boxes with b balls. Since by Lemma 4.3.8 we have the number of ways to fit partitions $\lambda \vdash b$ under the diagonal in an $n \times n$-box as in Figure 4.1, we need only sum over all $\lambda \vdash b$ to obtain the number of generators of the b-th graded piece $(\mathcal{A}_n)_b$. □

Corollary 4.3.10. *The Hilbert series of $\mathcal{A}_n = S/I_n$ can be written as*

$$\mathrm{Hilb}_S(\mathcal{A}_n, q) = \sum_{1 \leq b \leq n} \sum_{\lambda \vdash b} \left(\prod_{\substack{1 \leq j \leq \#\lambda \\ s.t.\ \lambda_j \neq \lambda_{j+1}}} \binom{n - \lambda_j - \#\{\lambda_i \mid \lambda_i > \lambda_j\}}{\#\{\lambda_i \mid \lambda_i = \lambda_j\}} \cdot q^{\lambda} \right)$$

Proof. The Hilbert function $\mathrm{HilbF}_S((\mathcal{A}_n)_b) = \dim_{\Bbbk}(\mathcal{A}_n)_b$ so by Theorem 4.3.9, we obtain the coefficients in the Hilbert series. □

CHAPTER 5

A Gröbner basis and Hilbert series for I_Δ^G

5.1. A Hilbert series for I_n^Δ

As discussed, a Gröbner basis for the $\mathfrak{S}_n$-invariant ideal $I_n := (e_1(\mathbf{x}), \ldots, e_n(\mathbf{x}))$, and thus the coinvariant algebra $\mathcal{A}_n = \Bbbk[x_1, \ldots, x_n]/(e_1(\mathbf{x}), \ldots, e_n(\mathbf{x}))$, is well-known. Since we have the basis for $\mathcal{A}_n$, the Hilbert series is easily calculable as well Theorem 4.3.3 states that

$$\mathrm{Hilb}(\mathcal{A}_n; q) = [n]_q!.$$

Recall from section 2.1 that the Stanley-Reisner ring for a simplex Δ_n on n vertices is the free polynomial ring over a field $\Bbbk$ in n variables. From from section 2.4, we also know that the ideal generated by the universal system of parameters $(\theta_1, \ldots, \theta_d)$ is equal to the $\mathfrak{S}_n$-invariant ideal $(e_1(\mathbf{x}), \ldots, e_n(\mathbf{x}))$. Thus,

$$\begin{aligned}
\Bbbk[\Delta_n]/(\theta_1, \ldots, \theta_d) &= \Bbbk[x_1, \ldots, x_n]/(\theta_1, \ldots, \theta_d) \\
&= \Bbbk[x_1, \ldots, x_n]/(e_1(\mathbf{x}), \ldots, e_n(\mathbf{x})) \\
&= \mathcal{A}_n.
\end{aligned}$$

It seems somewhat natural to then extend these results to any simplicial complex Δ. If Δ is a simplicial complex with minimal non-faces $G_1, \ldots, G_k$, then the Stanley-Reisner ideal for Δ is $I_\Delta = (m_1, \ldots, m_k)$ where $m_i = \mathbf{x}^{G_i}$. Note that each m_i is a squarefree monomial. This gives us the following equality of rings:

$$\begin{aligned}
\Bbbk[\Delta]/(\theta_1, \ldots, \theta_d) &= \Bbbk[x_1, \ldots, x_n]/I_\Delta + (\theta_1, \ldots, \theta_d) \\
&= \Bbbk[x_1, \ldots, x_n]/(\theta_1, \ldots, \theta_d, m_1, \ldots, m_k) \\
&= \Bbbk[x_1, \ldots, x_n]/(e_1(\mathbf{x}), \ldots, e_n(\mathbf{x}), m_1, \ldots, m_k).
\end{aligned}$$

Ultimately, we would like to find the Hilbert series for $\Bbbk[\Delta]/I_n^\Delta$ for any arbitrary I_Δ, however that problem at this point seems just beyond reach. In this section we focus on showing that this new method works for $I_\Delta = (m)$ where m is a squarefree monomial, i.e., the simplicial complex Δ has a single minimal non-face. We hope that with some more work, this method will generalize.

We begin by observing that when I_Δ is empty, e.g., Δ is an n-simplex, $\mathcal{A}_n^\Delta = \mathcal{A}_n$ and the theory holds from Section 4.3. The next base case is for I_Δ to be generated by a single squarefree monomial, $m = x_{i_1} \cdots x_{i_t}$, for a minimal non-face $G = \{i_1, \ldots, i_t\}$ of Δ. First, we intend to find the Hilbert series of I_n^Δ. The Hilbert series only cares about the dimension of the graded pieces of $\mathcal{A}_n^\Delta = S/(I_\Delta + I_n)$, so we consider the map that sends

$$x_{i_1} \cdots x_{i_t} \mapsto x_n \cdots x_{n-t}.$$

Then we need only find the dimension of each graded piece for $S/(I_n + (x_{n-t} \cdots x_n)$. Furthering the combinatorial model from Section 4.3, we notice that all monomials in $S = \Bbbk[x_1, \ldots, x_n]$ whose support is the set is a subset of $\{n-t, \ldots, n\}$ is divisible by m and therefore will not be a part of the basis of $\mathcal{A}_n^\Delta$.

We may compute these terms combinatorially, following the method from Section 4.3.

(1) Begin with the $n \times n$-box with upper diagonal boxes filled as in Figure 4.1.
(2) Let m^c be a distinct color. Fill in all the boxes in rows $n, \ldots, n-t$ with the color m^c.
(3) The rules for acceptable monomials are the same as before with one exception:
    (a) One begins by placing a ball in any box beginning from the left-most boarder.
    (b) Only one ball may be placed in each box.
    (c) For $\mathbf{x}^\mathbf{a}$, $\mathbf{a} = \#\{\text{boxes in row } i\}$.
    (d) The number of balls in the left-most column with color m^c cannot be greater than $\deg(m)$.

From these choices of balls in boxes, we may write down an integer partition. Let b be the number of total balls placed then $\lambda \vdash b$ is the partition where each part λ_j is equal to the number of balls in row j. We note that these may not be increasing, so we rearrange accordingly. Let $\mathbf{x}^\mathbf{a}$, where $a_i = \#\{\text{balls in row } i\}$, then we say $\mathbf{a}$ is equivalent to a partition λ if $\mathbf{a}$ and λ differ by rearrangement and a monomial is of type λ if its exponent vector is equivalent to λ. This format allows us to count the number of monomials of each type λ whose parts lie under the diagonal. A part λ_j of a partition $\lambda \vdash b$, $b \in \mathbb{N}$ has color m^c if λ_j is in a row which has boxes filled with color m^c.

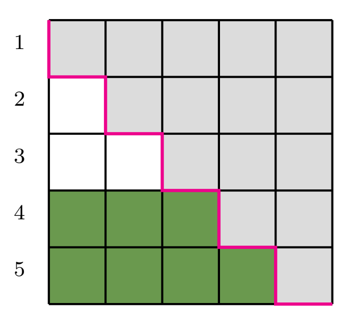
Theorem 5.1.1. *Let I_Δ be generated by a single squarefree monomial. Then*

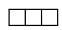
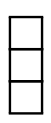
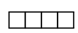
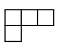
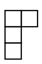
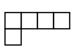
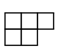
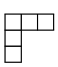
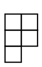
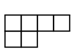
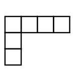
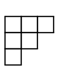
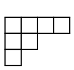
$$\mathrm{Hilb}(\mathcal{A}_n^\Delta, q) = \sum_{b\in\mathbb{N}} \sum_{\substack{\lambda\vdash b \\ s.t.\ Q(\lambda)}} q^\lambda$$

where $Q(\lambda)$ is the criterion that (1) if $\lambda = (\lambda_1, \ldots, \lambda_k)$, then all λ_i fit left-aligned under the diagonal, (2) $(n-1) - |\lambda| \geq \deg(m)$, (3) the number of parts of λ with color m^c must be less than $\deg(m)$.

Proof. Let $b \in \mathbb{N}$ and let $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash b$ such that λ fulfills the criterion $Q(\lambda)$. We will show that any monomial of type λ is in the generating set of the b-th graded piece of $\mathcal{A}_n^\Delta$. Say $\mathbf{x^a}$ is a monomial of type λ. This means that $\mathbf{a} = (a_1, \ldots, a_n)$ where $a_i = \#\{\text{balls in row } i\}$ and the non-zero a_i can be rearranged to equal λ. Since λ fulfils the first criterion, then $\mathbf{x^a}$ is a substaircase monomial by construction. Now we only need to show that it is not divisible by m. A monomial is divisible by m if it is of type μ where $(n-1) - |\mu| < \deg(m)$, thus any monomial that is in the basis for $\mathcal{A}_n^\Delta$ must be of type μ where μ is such that $(n-1) - |\mu| \geq \deg(m)$. Since λ fulfils (2), then it's possible that m does not divide $\mathbf{x^a}$. The only case left to check is that $\mathrm{supp}(m) \nsubseteq \mathrm{supp}(\mathbf{x^a})$. But λ fulfils criterion (3). Therefore, m does not divide $\mathbf{x^a}$ which is of degree b since the non-zero parts of $\mathbf{a}$ rearrange to equal λ. Thus, $\mathbf{x^a}$ is a generator of $(\mathcal{A}_n^\Delta)_b$. □

Example 5.1.2. *Let $n = 5$ and let $I_\Delta = (x_4x_5)$. The next figure demonstrates the final result after drawing a 5×5-box, drawing the diagonal. Here the blue represents the monomial x_4x_5.*



b	*Parititons*	*Configurations*	*# of choices*	*Total # of choices*
0	∅	1	1	1
1		5 4 3 2	4	4
2		5 4 3	1	
		$\substack{5\\3}\ 2\ \substack{4\\3}\ 2\ \substack{3\\2}$	5	8
3		5 4	2	
		$\substack{5\\3}\ 2\ \substack{4\\3}\ 2\ \substack{3\\5}\ 4\ 2$	7	
		$\substack{5\\3\\2}\ \substack{4\\3\\2}$	2	11
4		5	1	
		$\substack{5\\3}\ 2\ \substack{4\\3}\ 2$	4	
		$\substack{5\\3}\ \substack{4\\3}$	2	
		$\substack{5\\3\\2}\ \substack{4\\3\\2}\ \substack{3\\5\\2}\ \substack{4\\2}$	4	11
5		$\substack{5\\3}\ 2$	2	
		$\substack{5\\3}\ \substack{4\\3}$	2	
		$\substack{5\\3\\2}\ \substack{4\\3\\2}$	2	
		$\substack{5\\3\\2}\ \substack{4\\3\\2}$	2	8
6		$\substack{5\\3}$	1	
		$\substack{5\\3\\2}$	1	
		$\substack{5\\3\\2}\ \substack{4\\3\\2}$	2	4
7		$\substack{5\\3\\2}$	1	1

For the sake of space, we have removed all computations that contribute 0 *to the number of choices and leave it to the reader to confirm there are no possible ways to fit the parts of* λ *left-aligned under the diagonal without at least two parts being the color of* $x_4 * x_5$.
The Hilbert series of $\mathcal{A}_5^{\Delta}$ *is*

$$\mathrm{Hilb}(\mathcal{A}_5^{\Delta};q) = 1 + 4q + 8q^2 + 11q^3 + 11q^4 + 3q^5 + 4q^6 + q^7$$

Remark 5.1.3. *Before closing this section, we want to make some notes about generalizing this concept. We note that the proof of Theorem 5.1.1 only works because we are removing a single monomial. Removing any other monomials results in more complicated division as seen by the Buchberger algorithm.*

5.2. A Gröbner basis for I_n^{Δ}

Following from our knowledge for a basis when $\Delta = \Delta_n$ (an n-simplex), we ask for a generalization of this basis. However, as remarked earlier, this is a complicated request. In this section, we give a conjectural reduced Gröbner basis for the ideal $I_n^{\Delta} = (e_1(\mathbf{x}), \ldots, e_n(\mathbf{x}), m)$, i.e., Δ has exactly one minimal nonface. We denote this ideal I_n^{Δ} since it is the ideal I_{Δ} invariant under the stabilizer of m under the $\mathfrak{S}_n$ action. It is our belief that once a reduced Gröbner basis is known in the case when $I_{\Delta} = (m_1, m_2)$, then we will be able to write down a basis for I_n^{Δ} when I_{Δ} is generated by any arbitrary number of squarefree monomial ideals.

Definition 5.2.1. *For a squarefree monomial* $m \in \Bbbk[x_1, \ldots, x_n]$, *the* jump set *of* m *is the set* $\{k_1, \ldots, k_{\ell+1}\}$ *where* $x_{i_{k_j+1}} \cdots x_{i_{k_{j+1}}}$ *has the property that for all* $k_j + 1 \leq m \leq k_{j+1}$, $i_{m+1} = i_m + 1$, *i.e., the sequence* $(i_{k_j+1}, \cdots, i_{k_{j+1}})$ *increases by exactly one at each step. If* $i_1 \neq 1$, *then we call the jump from* 1 *to* i_1 *an* open jump. *Similarly, if* $i_t \neq n$, *then we call the jump from* i_t *to* n *an* open jump. *Any other jump* (i_{k_j}, i_{k_j+1}) *is called a* closed jump.

Since any square free monomial can be written in the form

$$m = x_{i_1} \cdots x_{i_{k_1}} x_{i_{k_1+1}} \cdots x_{i_{k_2}} x_{i_{k_2+1}} \cdots\cdots x_{i_{k_\ell}} x_{i_{k_\ell+1}} \cdots x_{i_t},$$

then a jump set uniquely determines a squarefree monomial. For ease of notation, we partition the support of m into sets $K_1, \ldots, K_{\ell+1}$, i.e., following the form in Definition 5.2.1, $K_j =$

$\{i_{k_{j+1}}, \cdots, i_{k_j+1}\}$ so that

$$m = \underbrace{x_{i_1} \cdots x_{i_{k_1}}}_{\mathbf{x}^{K_1}} \underbrace{x_{i_{k_1+1}} \cdots x_{i_{k_2}}}_{\mathbf{x}^{K_2}} x_{i_{k_2+1}} \cdots\cdots x_{i_{k_\ell}} \underbrace{x_{i_{k_\ell+1}} \cdots x_{i_t}}_{\mathbf{x}^{K_{\ell+1}}}.$$

Conjecture 5.2.2 (Gröbner basis for $I_n^\Delta = (e_1(\mathbf{x}_n), \ldots, e_n(\mathbf{x}_n), m)$). *Let $S = \Bbbk[x_1, \ldots, x_n]$ be a polynomial ring over a field $\Bbbk$. Let $e_1(\mathbf{x}_n), \ldots, e_n(\mathbf{x}_n)$ be the elementary symmetric functions in n variables. Then for any squarefree monomial $m \in S$, the reduced Gröbner basis for $I_n^\Delta = (e_1(\mathbf{x}_n), \ldots, e_n(\mathbf{x}_n), m)$ is given by the following list of polynomials:*

$h_1[x_1, \ldots, x_n], \ldots, h_p[x_p, \ldots, x_n]$,

$\left(\prod_{j=2}^{\ell+1}(x_{i_{k_j+1}} \cdots x_{i_{k_{j+1}}})\right) h_{i_{k_1}}[x_{i_{k_1}+1}, \ldots, x_n], \ldots, \left(\prod_{j=2}^{\ell+1}(x_{i_{k_j+1}} \cdots x_{i_{k_{j+1}}})\right) h_{i_{k_1+1}-2}[x_{i_{k_1+1}-1}, \ldots, x_n]$,

$\left(\prod_{j=2}^{\ell+1}(x_{i_{k_j+1}} \cdots x_{i_{k_{j+1}}})\right) h_{i_{k_2}}[x_{i_{k_2}+1}, \ldots, x_n], \ldots, \left(\prod_{j=2}^{\ell+1}(x_{i_{k_j+1}} \cdots x_{i_{k_{j+1}}})\right) h_{i_{k_2+1}-2}[x_{i_{k_2+1}-1}, \ldots, x_n], \ldots,$

$\left(\prod_{j=\ell+1}^{\ell+1}(x_{i_{k_j+1}} \cdots x_{i_{k_{j+1}}})\right) h_{i_{k_\ell}}[x_{i_{k_\ell}+1}, \ldots, x_n], \ldots, \left(\prod_{j=\ell+1}^{\ell+1}(x_{i_{k_j+1}} \cdots x_{i_{k_{j+1}}})\right) h_{i_{k_\ell+1}-2}[x_{i_{k_\ell+1}-1}, \ldots, x_n]$

$x_{i_t} h_{i_t-1}[x_{i_t}, \ldots, x_n]$,

$h_{i_t}[x_{i_t+1}, \ldots, x_n], \ldots, h_{n-1}[x_n]$.

In certain circumstances, the following adjustments are made to the basis:

- *if $p = i_{k_1}$, then the basis also includes*

$$h_{i_{k_1}+1}[x_{i_{k_1}+1}, \ldots, \widehat{x_{i_t}}, \ldots, x_n], \ldots, h_{i_t-1}[x_{i_t-1}, \ldots, \widehat{x_{i_t}}, \ldots, x_n].$$

- *if $i_{k_1} > 1$, then m itself is included in the basis as well.*
- *furthermore, when $i_{k_1} \geq t$, we substitute $h_t[x_t, \ldots, x_n]$ with $h_t[x_t, \ldots, x_n] - m$.*

Here,

- *$\{k_1, \ldots, k_t\}$ is the jump set of m*
- *$p = i_{k_1}$ when $i_t < n$ and $p = i_t$ when $i_t = n$;*

- *and $\widehat{x_{i_t}}$ denotes the removal of x_{i_t} from the set of variables in which the reduced homogeneous function $h_m[x_m, \ldots, x_n]$ is generated.*

Since I_n^{Δ} is a homogeneous ideal generated by a Gröbner basis, then the Gröbner basis is also homogeneous. Therefore, the proof will follow from counting the number of elements of each degree and comparing them with the Hilbert series of Theorem 5.1.1 and then showing they are in the ideal.

The following Corollary follows by observing the degrelex ordering on the polynomials listed in Conjecture 5.2.2.

Corollary 5.2.3. *The leading term ideal $\mathrm{LT}(I_n^{\Delta})$ is generated by the monomials*

$$
\begin{aligned}
&x_1, x_2^2, \ldots, x_p^p,\\
&\left(\prod_{j=2}^{\ell+1} \mathbf{x}^{K_j}\right) x_{i_{k_1}+1}^{i_{k_1}}, \ldots, \left(\prod_{j=2}^{\ell+1} \mathbf{x}^{K_j}\right) x_{i_{k_1+1}-1}^{i_{k_1+1}-2},\\
&\ldots\\
&\mathbf{x}^{K_{\ell+1}} x_{i_{k_\ell}+1}^{i_{k_\ell}}, \ldots, \mathbf{x}^{K_{\ell+1}} x_{i_{k_\ell+1}-1}^{i_{k_\ell+1}-2}\\
&x_{i_t}^{i_t}, x_{i_t+1}^{i_t}, \ldots, x_n^{n-1}.
\end{aligned}
$$

- *If $p = i_{k_1}$, then we also have*

$$x_{i_{k_1}+1}^{i_{k_1}+1}, \ldots, x_{i_t-1}^{i_t-1}.$$

- *If $i_{k_1} > 1$, then m is also a leading term.*

5.3. SageMath Code

The conjectural basis in Conjecture 5.2.2 has been verified for many various squarefree monomials in polynomial rings with up to $n = 10$ variables. We provide the code for future verification and expansion. The example in the code is for the simplicial complex on n vertices with the minimal nonface $G = \{1, 2, 6\}$. Note that in SageMath the counting begins at 0 instead of 1. It is important for our cases and future generalizations, that we translate the counting back to beginning at 1.

```
R.<x0,x1,x2,x3,x4,x5,x6,x7,x8> = QQ['x0','x1','x2','x3','x4','x5','x6','x7','x8']
e = SymmetricFunctions(QQ).e()
n = 9
m = x0*x1*x5
L = [m]
for i in range(n):
   ei = e[i+1].expand(n)
   L.append(ei)
J = ideal(L);
BJ = J.groebner_basis();
sep = ''
for i in range(len(Sequence(BJ))):
    pretty_print(Sequence(BJ)[i])
    sep
```

CHAPTER 6

The search for an equivariant isomorphism of Stanley-Reisner rings

6.1. An equivariant Tor calculation for $\Bbbk[\Delta]$

Although it is our goal to eventually give an explicit formula for $\mathrm{Tor}_i^A(\Bbbk[\Delta], \Bbbk)$ so that it can be compared to $\mathrm{Tor}_i^A(\Bbbk[\mathrm{Sd}\Delta], \Bbbk)$ in order to prove or disprove Conjecture 3.3.4 or more weakly Conjecture 3.3.2, here we apply our knowledge of the Hilbert series for $\mathcal{A}_n^\Delta$ to compute the dimensions of $\mathrm{Tor}_0^A(\Bbbk[\Delta], \Bbbk)_j$.

LEMMA 6.1.1. *Let Δ be a simplicial complex on n vertices with Stanley-Reisner ring $\Bbbk[\Delta] = S/I_\Delta$ and let $I_n = (e_1(\mathbf{x}), \ldots, e_n(\mathbf{x}))$. If we write $I_n^\Delta = I_n + I_\Delta$, then*

$$\mathrm{Tor}_0^A(\Bbbk[\Delta], \Bbbk) \cong \Bbbk[x_1, \ldots, x_n]/I_n^\Delta.$$

PROOF. A basic property of Tor gives us that

$$\mathrm{Tor}_0^A(\Bbbk[\Delta], \Bbbk) = \Bbbk[\Delta] \otimes_A \Bbbk.$$

Since $\Bbbk$ is an $A = \Bbbk[z_1, \ldots, z_d]$-module by evalutation map $A \to \Bbbk[\Delta]$, $z_j \mapsto \varepsilon_j(\mathbf{x}) \mapsto 0$, then

$$\begin{aligned} \Bbbk[\Delta] \otimes_A \Bbbk &\cong \Bbbk[\Delta]/I_n \\ &= \Bbbk[x_1, \ldots, x_n]/(I_n + I_\Delta) \\ &= \Bbbk[x_1, \ldots, x_n]/I_n^\Delta. \end{aligned}$$

□

THEOREM 6.1.2. *Let Δ be a simplicial complex with a single minimal non-face. Then*

$$\dim_\Bbbk \mathrm{Tor}_0^A(\Bbbk[\Delta])_b = \#\{\lambda \vdash b \mid Q(\lambda)\},$$

where $Q(\lambda)$ is the criterion that (1) if $\lambda = (\lambda_1, \ldots, \lambda_k)$, then all λ_i fit left-aligned under the diagonal, (2) $(n-1)-|\lambda| \geq \deg(m)$, and (3) the number of parts of λ with color m^c must be less than $\deg(m)$.

Proof. By Theorem 5.1.1 we are given the Hilbert series of $\mathcal{A}_n^{\Delta}$. The result follows from the isomorphism from Lemma 6.1.1. ☐

6.2. Future work

Since the Hilbert series of $\mathcal{A}_n^{\Delta} = S/(I_n + I_{\Delta})$ is invariant under the automorphism taking a $x_{i_1} \cdots x_{i_t} \in I_{\Delta}\ \mathfrak{S}_n(m) \mapsto x_{n-t} \cdots x_n$, then the Hilbert series for $\mathcal{A}_n^{\Delta}$ depends only on the number of monomials of each degree in I_{Δ} and not on the particular choices for support sets for each monomial in I_{Δ}. It is our hope to use this to extend our results to be able to write down the Hilbert series of $\mathcal{A}_n^{\Delta}$ for an arbitrary choices of I_{Δ}. So although we will not be able to write down the particular syzygies, we will be able to explicitly compute the $\mathbb{N}$-graded Betti numbers for the 0th syzygy module in the resolution of $\Bbbk[\Delta]$ as an A-module. Once we have these counts, our desire is to then be able to compute the rest of the Betti numbers using results from commutative algebra. We also hope to then be able to compare them with the Betti numbers for the resolution of $\Bbbk[\mathrm{Sd}\Delta]$ over A as given by Theorem 3.2.5. Finally, we hope to use a similar technique used to obtain the basis given in Conjecture 5.2.2 to be able to find a basis for $\mathcal{A}_n^{\Delta}$ in total generality. As there is a clear pattern and as they are a subset of the substaircase monomials, there may be a way to do this combinatorially using the "balls in boxes" method described in this paper.

6.3. Big Example

Let Δ be a simplicial complex on $n = 4$ vertices with facets $F_1 = \{1, 2, 4\}$ and $F_2 = \{2, 3, 4\}$ The Stanley-Reisner ring of Δ is

$$\Bbbk[\Delta] = S/I_{\Delta} = \Bbbk[x_1, x_2, x_3, x_4]/(x_1x_3).$$

From Conjecture 5.2.2 (verified with `SageMath` and `Macaulay2`) the basis for the coinvariant Stanley-Reisner ring

$$\mathcal{A}_n^{\Delta} = \Bbbk[\Delta]/(e_1(\mathbf{x}), e_2(\mathbf{x}), e_3(\mathbf{x}), e_4(\mathbf{x}), x_1x_3)$$

is

$$\{x_1, x_2x_3, x_2^2, x_3^3, x_4^3\}.$$

This means that we can calculate the basis elements for each graded piece of $\mathcal{A}_n^{\Delta}$.

Degree j	Spanning Sets of $(\mathcal{A}_n^{\Delta})_j$	Resolution Factor
0	$\{1\}$	$A(-0)^1$
1	$\{x_2, x_3, x_4\}$	$A(-1)^3$
2	$\{x_3^2, x_4^2, x_2x_4, x_3x_4\}$	$A(-2)^4$
3	$\{x_2x_4^2, x_3^2x_4, x_3x_4^2\}$	$A(-3)^3$
4	$\{x_3^2x_4^2\}$	$A(-4)^1$

Since Δ is Cohen-Macaulay, there is no higher homology. Thus, the resolution of $\Bbbk[\Delta]$ as an A-module is

$$\mathcal{F}_\bullet^{\Delta}: \quad 0 \longleftarrow \Bbbk[\Delta] \xleftarrow{\partial_0} \begin{array}{c} A(-0)^1 \\ \oplus \\ A(-1)^3 \\ \oplus \\ A(-2)^4 \\ \oplus \\ A(-3)^3 \\ \oplus \\ A(-4)^1 \end{array} \longleftarrow 0.$$

The Barycentric subdivision of Δ, denoted $\mathrm{Sd}\Delta$, is given by facets:

$$\{1, 12, 124\}, \{1, 14, 124\},$$
$$\{2, 12, 124\}, \{2, 24, 124\},$$
$$\{4, 14, 124\}, \{4, 24, 124\},$$
$$\{2, 23, 234\}, \{2, 24, 234\},$$
$$\{3, 23, 234\}, \{3, 34, 234\},$$
$$\{4, 24, 234\}, \{4, 34, 234\}.$$

The Stanley-Reisner ring a la Definition 3.1.1, is

$$\Bbbk[\mathrm{Sd}\Delta] = \Bbbk[y_1, y_2, y_3, y_4, y_{12}, y_{14}, y_{23}, y_{24}, y_{34}, y_{124}, y_{234}]/I_{\mathrm{Sd}}\Delta,$$

where

$$\begin{aligned} I_{\mathrm{Sd}}\Delta = (&y_1y_2, y_1y_3, y_1y_4, y_2y_3, y_2y_4, y_3y_4,\\ &y_{12}y_{14}, y_{12}y_{23}, y_{12}y_{24}, y_{12}y_{34}, y_{14}y_{23}, y_{14}y_{24}, y_{14}y_{34}, y_{23}y_{24}, y_{23}y_{34}, y_{24}y_{34},\\ &y_{124}y_{234},\\ &y_1y_{23}, y_1y_{24}, y_1y_{34}, y_2y_{14}, y_2y_{34}, y_3y_{12}, y_3y_{14}, y_3y_{24}, y_3y_{24}, y_4y_{12}, y_4y_{23},\\ &y_1y_{234}, y_3y_{124}, y_{12}y_{234}, y_{14}y_{234}, y_{23}y_{124}, y_{34}y_{124}) \end{aligned}$$

When $\mathrm{Sd}\Delta$ is given a proper balanced coloring, then the colorful system of parameters, $\gamma = (\gamma_1, \gamma_2, \gamma_3)$, for $\Bbbk[\mathrm{Sd}\Delta]$ are

$$\gamma_1 = y_1 + y_2 + y_3 + y_4, \quad \gamma_2 = y_{12} + y_{14} + y_{23} + y_{24} + y_{34} \quad \gamma_3 = y_{124} + y_{234}$$

Since Δ is Cohen-Macaulay, $\mathrm{Sd}\Delta$ is also Cohen-Macaulay, which means $\mathrm{Sd}\Delta$ has no higher homology (dually, cohomology). By Theorem 3.2.5, we obtain the following resolution for $\Bbbk[\mathrm{Sd}\Delta]$ as an A-module:

$$\mathcal{F}_\bullet^{\mathrm{Sd}\Delta}: \quad 0 \longleftarrow \Bbbk[\mathrm{Sd}\Delta] \xleftarrow{\partial_0} \begin{matrix} A(-0)^1 \\ \oplus \\ A(-1)^3 \\ \oplus \\ A(-2)^4 \\ \oplus \\ A(-3)^3 \\ \oplus \\ A(-4)^1 \end{matrix} \longleftarrow 0.$$

We notice that $\mathcal{F}^{\mathrm{Sd}}\Delta_\bullet$ has the same graded Betti numbers as $\mathcal{F}_\bullet^\Delta$. Therefore, in this case, Conjecture 3.3.2. We want to note that this is not entirely new information. We already did know that the conjecture held for Cohen-Macaulay simplicial complexes. However, the constructive method of obtaining the equality by computing the Betti numbers directly from the basis of $\mathcal{A}_n^\Delta$ is new.

Bibliography

[1] A. Adams and V. Reiner, *A colorful hochster formula and universal parameters for face rings*, Journal of Commutative Algebra, 15.2 (2023), pp. 151–176.

[2] E. Artin and A. N. Milgram, *Galois theory*, vol. 2, Courier Corporation, 1998.

[3] F. Bergeron, *Algebraic combinatorics and coinvariant spaces*, CRC Press, 2009.

[4] V. M. Buchstaber and T. E. Pano, *Toric topology*, vol. 204, American Mathematical Soc., 2015.

[5] C. de Concini, D. Eisenbud, and C. Procesi, *Hodge algebras*, Astérisque, 91 (1982), pp. 1261–1277.

[6] D. Eisenbud, *Commutative algebra: with a view toward algebraic geometry*, vol. 150, Springer Science & Business Media, 2013.

[7] G. Fløystad, J. McCullough, and I. Peeva, *Three themes of syzygies*, Bulletin of the American Mathematical Society, 53 (2016), pp. 415–435.

[8] A. M. Garsia, *Combinatorial methods in the theory of cohen-macaulay rings*, Advances in Mathematics, 38 (1980), pp. 229–266.

[9] A. M. Garsia and D. Stanton, *Group actions on stanley-reisner rings and invariants of permutation groups*, Advances in Mathematics, 51 (1984), pp. 107–201.

[10] M. Hochster, *Cohen-macaulay rings, combinatorics, and simplicial complexes, in" ring theory ii"*, Lect. Notes in Pure Appl. Math., (1977), pp. 171–223.

[11] J. Huang, *0-hecke algebra action on the stanley-reisner ring of the boolean algebra*, Annals of Combinatorics, 19.2 (2015), pp. 293–323.

[12] S. M. Jürgen Herzog, *Systems of parameters and the cohen–macaulay property*, Journal of Algebraic Combinatorics, 54.4 (2021), pp. 1261–1277.

[13] E. Miller and B. Sturmfels, *Combinatorial commutative algebra*, vol. 227, Springer, 2005.

[14] D. Smith, *On the cohen-macaulay property in commutative algebra and simplicial topology*, Pacific Journal of Mathematics, 141 (1990), pp. 165–196.

[15] R. P. Stanley, *Combinatorics and commutative algebra*, vol. 41, Springer Science & Business Media, 2007.

[16] B. Sturmfels, *Grobner bases and convex polytopes*, vol. 8, American Mathematical Soc., 1996.

[17] ——, *Algorithms in invariant theory*, Springer Science & Business Media, 2008.