The Mellin transform and its properties
A Paper for MAT 501: Complex Analysis

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1 Introduction

The Mellin transform is the complex valued holomorphic function

$$\mathcal{M}[f(x);s] := f^*(s) = \int_0^\infty f(x)x^{s-1}dx$$

defined over the positive reals was first used by Riemann and Cahen and then set to rigor by Hjalmar Mellin. It is common to call f and f^* a *Mellin pair.* By simple derivations, we may obtain the Mellin transform from both the Laplace transform and the Fourier transform through relatively rudimentary calculations found in sections 2.2 and 2.3, respectively.

Despite the simple correspondence to the Laplace and Fourier transforms, the Mellin transform is a powerful tool in its own right. In particular, it can be useful to apply the Mellin transform to a harmonic function in order to analyze the asymptotics of $F(x)$ as $x \to 0$ or $x \to \infty$. A *harmonic function* is function of the form

$$F(x) = \sum_n \lambda_n f(\mu_n x),$$

where we call λ_n the *amplitude*, μ_n the *frequency*, and $f(x)$ the *base function.* This process results in a separation of the amplitude-frequency pair, (λ, μ), from the base function. We will remark here that the output of $F(x)$ is often called a *harmonic sum.* To factor out the amplitude-frequency pair, we require the *rescaling rule*, which states that for $\mu > 0$, $f(\mu x)$ implies $f^*(s) = \mu^{-s} f(x)$. Now by appealing to the linearity of the transform, we can explicitly write the Mellin transform of F as

$$F^*(s) = \sum_n (\lambda_n \mu_n^{-s}) \cdot f^*(s).$$

This form is particularly nice because the sum

$$\Lambda(s) = \sum_n (\lambda_n \mu_n^{-s})$$

is a generalized Dirichlet series. We will not go into depth about the Dirichlet series within this paper. For more background see [1].

In this paper we will will begin by covering basic properties of the Mellin transform and then cover the two important properties which give it such power:

1. harmonic functions and the separation property (section **??**)
2. and the mapping property (section 3.)

A more extensive list of uses can be found in [2].

2 Basic Properties

2.1 Functional Properties

Let $f(x)$ be a function whose Mellin transform, $f^*(s)$, has the fundamental strip $\langle -\alpha, -\beta \rangle$. Then the following properties hold for f:

(i) *Linearity:* $\lambda f(x) \mapsto \lambda f^*(s)$.

(ii) *Rescaling rule:* $f(\mu x) \mapsto \mu^{-s} f^*(s)$, for $\mu > 0$.

(iii) *Power rule 1:* $x^\nu f(x) \mapsto f^*(s+\nu)$, $\langle -\alpha - \nu, -\beta - \nu \rangle$.

(iv) *Power rule 2:* $f(x^\rho) \mapsto \frac{1}{\rho} f^*(\frac{s}{\rho})$, $\langle -\rho\alpha, -\rho\beta \rangle$.

2.2 As derived from the Laplace transform

The Laplace transform of a function g is

$$\mathcal{L}[g(t); s] := \int_{-\infty}^{+\infty} g(t) e^{-st} dt$$

for $s \in \mathbb{C}$.

By then setting $t = -\log(x)$ and $g(-\log(x)) = f(x)$, we obtain the Mellin transform from the Laplace transform:

$$\begin{aligned} \mathcal{L}[g(-\log(x)); s] &= \int_0^{+\infty} g(-\log(x)) e^{-s(-\log(x))} \frac{dx}{x} \\ &= \int_0^{+\infty} f(x) x^{s-1} dx \end{aligned}$$

2.3 As derived from the Fourier transform

The Fourier transform of a function g is

$$\mathcal{F}[g(t); \zeta] := \int_{-\infty}^{+\infty} g(t) e^{-2\pi\zeta t} dt.$$

To obtain the Mellin transform from $\mathcal{F}$ we apply the substitution $\zeta = \frac{s-c}{2\pi i}$ with $c \geq 0$ to

$$\begin{aligned} \mathcal{F}[g(e^t); \zeta] &= \int_{-\infty}^{+\infty} f(e^t) e^{2\pi i \zeta t} dt \\ &= \int_{-\infty}^{+\infty} f(e^t) e^{-(s-c)t} dt \\ &= \int_0^{+\infty} f(x) e^{-(s-c)\log(x)} \frac{dx}{x} && \text{(Set } e^t = x\text{)} \\ &= \int_0^{+\infty} g(x) x^{-c} x^s \frac{dx}{x} \\ &= \int_0^{+\infty} f(x) x^{s-1} dx && \text{(Set } f(x) = g(x)x^{-c}\text{)} \end{aligned}$$

2.4 The Fundamental Strip

When we derived the Mellin transform from the Leplace and Fourier transforms above we put no constraints upon the complex parameter s, effectively deceiving the reader. Upon closer inspection, one may notice that $f^*(s)$, does not exist everywhere. If

$$f(x) = \begin{cases} \mathcal{O}(x^{\alpha}) & \text{as } x \to 0, \\ \mathcal{O}(x^{\beta}) & \text{as } x \to +\infty, \end{cases} \tag{1}$$

and is continuous on $(0, +\infty)$, then

$$\begin{aligned} \left| \int_0^{\infty} f(x) x^{s-1} dx \right| &\leq \int_0^{\infty} |f(x)| x^{s-1} dx \\ &= \int_0^{1} |f(x)| x^{s-1} dx + \int_1^{+\infty} |f(x)| x^{s-1} dx \\ &\leq \int_0^{1} |f(x)| x^{\Re(s)-1} dx + \int_1^{+\infty} |f(x)| x^{\Re(s)-1} dx. \end{aligned}$$

If c_1 is a constant, then

$$\int_0^1 |f(x)| x^{\Re(s)-1} dx \leq c_1 \int_0^1 x^{\Re(s)+\alpha-1} dx$$

exists for $\Re(s) > -\alpha$. Furthermore, if c_2 is also a constant, then

$$\int_1^{\infty} |f(x)| x^{\Re(s)-1} dx \leq c_2 \int_1^{\infty} x^{\Re(s)+\beta-1} dx.$$

exists for $\Re(s) < -\beta$. Thus, the Mellin transform must exist for $-\alpha < \Re(s) < -\beta$. We denote this interval as $\langle -\alpha, -\beta \rangle$ and call it the *fundamental strip.* Since f^* is analytically dependent on s, then f^* is analytically continuous within its fundamental strip. To summarize, the conditions we put on $f(x)$ in equation 1 guarantee the existence of $f^*(s)$ for $s \in \langle -\alpha, -\beta \rangle$ with $\alpha < \beta$.

Example 2.1. *Perhaps the most important examples is when we let $f(x) = e^{-x}$, whose Mellin transform,*

$$f^*(s) = \int_0^{\infty} e^{-x} x^{s-1} dx,$$

can be easily recognized as the Gamma function, $\Gamma(s)$. To find the fundamental strip, we refer to the properties of f defined in equation 1 and observe that (1),

$$f(x) \sim 1 \text{ as } x \to 0 \qquad \text{and} \qquad f(x) = \mathcal{O}(x^{-\beta}), \text{ for } \beta > 0 \text{ as } x \to +\infty.$$

Thus, $f^*(s)$ has the fundamental strip $\langle 0, +\infty \rangle$.

Example 2.2. *We now consider another example related to the Gamma function by the identity*

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{sin \pi s}.$$

The base function

$$f(x) = \frac{1}{1+x} = \begin{cases} \mathcal{O}(x^0) & \text{as } x \to 0 \\ \mathcal{O}(x^{-1}) & \text{as } x \to \infty. \end{cases}$$

So the Mellin transform,

$$f^*(x) = \int_0^{\infty} \frac{1}{1+x} x^{s-1} dx = \int_0^1 t^s (1-t)^{1-s} dt = \Gamma(s)\Gamma(1-s) = \frac{\pi}{sin \pi s},$$

has the fundamental strip $\langle 0, 1 \rangle$.

2.5 Inverse Mellin transform

Since the Mellin transform is so closely related to the Fourier and Laplace transforms it is natural that there should be an inversion formula.

Theorem 2.3 (Mellin Inversion Formula). *Let $f(x)$ be a continuous function on $(0,\infty)$ with Mellin transform $f^*(s)$ analytic in the interval $\langle -\alpha, -\beta\rangle$ and let $-\alpha < c < -\beta$ such that $f^*(c+it)$ integrable, then*

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f^*(s)x^{-s}ds \tag{2}$$

is the inverse Mellin transform.

The proof follows from reversing our computation in section 2.3 and then computing the inverse Mellin formula from the inverse Fourier transform. The importance of the inverse Mellin formula will be discussed in the next section.

3 The Direct Mapping Property of the Mellin Transform

By further investigation into the inverse Mellin transform we are able to obtain valuable asymptotic expansions for complex integrable functions $f(x)$. The inverse Mellin integral can by estimated by residues as long as the continuation of $f^*(s)$ does not grow quickly along vertical lines. This can be referred to as the *smallness* of Mellin transforms.

3.1 The Smallness of the Mellin transform

If $s = \sigma + it$ and if f is both continuous and r times differentiable, then we apply the Riemann-Lebesgue lemma

$$f^*(\sigma + it) = o(1) \qquad \text{as } t \to \pm\infty.$$

Then, since

$$\mathcal{M}[f^{(r)}(x); s] = (-1)^r(s-r)^r f^*(s-r)$$

we

$$f^*(s) = f^*(\sigma + it) = o(|t|^{-r}) \qquad \text{as } t \to \pm\infty.$$

With all of the previous properties in mind, we finally introduce the direct mapping property that allows us to use the Mellin transform in powerful ways. The Mellin transform maps asymptotics of a function $f(x)$ near 0 and ∞ to singular terms in the singular expansion $f^*(s)$. Most of the following is taken from [2].

Theorem 3.1 (Direct Mapping). *Let f be a continuous function and let f^* have a non-empty fundamental strip $\langle \alpha, \beta\rangle$.*

1. *If as $x \to 0$ f has the form*

$$f(x) = \sum_{(\ell,k)} c_{\ell,k} x^{\ell}(\log x)^k + \mathcal{O}(x^{\gamma}) \tag{3}$$

for $k > 0$ and $-\gamma < -\ell \leq \alpha$, then $f^*(s)$ is continuable to the strip $\langle -\gamma, \beta\rangle$ and has singular expansion

$$f^*(s) \sim \sum_{(\ell,k)} c_{\ell,k}\frac{(-1)^k k!}{(s+\ell)^{k+1}}. \tag{4}$$

2. Similarly, if $x \to \infty$ and $f(x)$ has the form as above but with $\beta \leq -\ell < -\gamma$, then $\langle \alpha, -\gamma \rangle$

$$f^*(s) \sim -\sum_{(\ell,k)} c_{\ell,k} \frac{(-1)^k k!}{(s+\ell)^{k+1}}.$$

Proof. We first observe that the singular expansions of $f^*(s)$ are related by the functional property

$$\mathcal{M}[f((x)^{-1}; s] = \mathcal{M}[f(x); s]$$

so we only prove the case when $x \to 0$. We begin by setting

$$g(x) = f(x) - \sum_{(\ell,k)} c_{\ell,k} x^\ell (\log x)^k \tag{5}$$

so that $g(x) = \mathcal{O}(x^\gamma)$ as $x \to 0$. Using this equality and the definition of of $f^*(s)$ we split the integral so that

$$\begin{aligned}
f^*(s) &= \int_0^1 f(x) x^{s-1} dx + \int_1^\infty f(x) x^{s-1} dx \\
&= \int_0^1 \left(g(x) + \sum_{(\ell,k)} c_{\ell,k} x^\ell (\log x)^k \right) x^{s-1} dx + \int_1^\infty f(x) x^{s-1} dx \\
&= \int_0^1 g(x) x^{s-1} dx + \int_0^1 \sum_{(\ell,k)} c_{\ell,k} x^\ell (\log x)^k x^{s-1} dx + \int_1^\infty f(x) x^{s-1} dx.
\end{aligned}$$

We now have three integrals. The first integral and the last integral are analytic in the strips $\langle -\ell, +\infty \rangle$ and $\langle -\infty, \ell \rangle$ respectively. Then by simply integrating the middle integral, which we may do since it is meromorphic in all of $\mathbb{C}$ then we obtain the desired result

$$f^*(s) \sim \sum_{(\ell,k)} c_{\ell,k} \frac{(-1)^k k!}{(s+\ell)^{k+1}}$$

with $s \in \langle -\ell, \beta \rangle$. □

Remark 3.2. *We note that if $f(x) = x^\ell H(x)$ where $H(x) = [[0 \leq x \leq 1]]$ is the Heaviside function, then*

$$f^*(s) = \frac{1}{s+\ell} \qquad s \in \langle -\ell, \infty \rangle.$$

So then

$$H^*(s) = \frac{(-1)^k k!}{(s+\ell)^{k+1}} \qquad s \in \langle -1, 1 \rangle,$$

which we observe is the singular expansion in equation 4.

References

[1] Tom M Apostol. *Introduction to analytic number theory.* Springer Science & Business Media, 1998.

[2] Philippe Flajolet, Xavier Gourdon, and Philippe Dumas. Mellin transforms and asymptotics: Harmonic sums. *Theoretical computer science*, 144(1-2):3–58, 1995.