HANDLING A HANDLEBODY DECOMPOSITION OF A CLOSED MANIFOLD
MATH 5378: DIFFERENTIAL GEOMETRY

ASHLEIGH ADAMS

1. Introduction

Morse theory, introduced by Milton Morse in 1930 [3], allows us to use differential functions to understand the structure of a given manifold and compute topological invariants thereof. More specifically, there is a certain decomposition called a handlebody decomposition, which we will define in more rigor shortly, wherein we take a closed smooth manifold and "chop it up" into smaller blocks called cells or handles allowing us to understand the shapes of manifolds and its homology. In this paper we give the necessary background to ultimately show that a Morse function on a closed manifold gives rise to a handlebody decomposition.

2. Background & Lemmas

In this section we will give the necessary definitions and lemmas to prove the main theorem. Many of the definitions in this section come from [1, 4]. An m-dimensional manifold M (also referred to as an m-manifold) is a topological object where for each $p \in M$ there exists a local coordinate system about p that is homeomorphic to a Euclidean space of dimension m [6, 4]. In order to build the theory, we will assume that all finite dimensional manifolds are smooth and all manifolds are of dimension m unless stated otherwise. An m-manifold is called *smooth* (or of class C^∞) if all derivatives of all orders exist. When dealing with a smooth manifold M with boundary ∂M, one must be careful about the boundary, which is why we often look at the manifold without its boundary (i.e., the complement of ∂M) separately. The *interior* of a manifold, is a manifold without its boundary:

$$\text{int}(M) = M - \partial M.$$

If we look at each point $p \in \text{int}(M)$ there exists a smooth (C^∞) coordinate system $\mathbf{x} = (x_1, \ldots, x_m)$ about p. Similarly, for each $p \in \partial M$, there exists a smooth coordinate system $\mathbf{x} = (x_1, \ldots, x_m)$ if $x_m \geq 0$. [1]

A function $f : M \to \mathbb{R}$ is called *smooth* (C^∞) if all derivatives of all orders exist with respect to a coordinate system in M. We can be a bit more careful here with the definition of a smooth function. Either $p \in \text{int}(M)$ or $p \in \partial M$. Treating these cases separately we give the following definitions:

- Let $p \in \text{int}(M)$. Then f is smooth in M if it is smooth with respect to a local coordinate system in a sufficiently small neighborhood of p.
- Let $p \in \partial M$. Then f is called smooth in M if it is smooth with respect to a local coordinate system $\mathbf{x}$ with $x_m \geq 0$ in a sufficiently small neighborhood of p, if we can extend $f(\mathbf{x})$ to a function $\tilde{f}(y_1, \ldots, y_m)$ which is smooth with respect to a coordinate system $(y_1, \ldots, y_m)$ where $y_i \in \mathbb{R}$.

Let M be an m-manifold without boundary. Given any function $f : M \to \mathbb{R}$ we look at the points where something interesting occurs. We call these points, critical points. More specifically, if for a point $p_0 \in M$

$$\frac{\partial f}{\partial x_1}(p_0) = \frac{\partial f}{\partial x_2}(p_0) = \cdots = \frac{\partial f}{\partial x_m}(p_0) = 0$$

with respect to a local coordinate system $\mathbf{x}$ about p_0, then p_0 is called a *critical point of f*. The value $f(p_0) \in \mathbb{R}$ is called the *critical value.*

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[1]Note that if one wants a stronger statement, they may look up *collar neighborhoods* of a manifold with boundary.

Critical points have two characterizations: degenerate and non-degenerate. In the most simple case, that is, the case when $M = \mathbb{R}^2$ and we have f defined on M as a single variable function so that $y = f(x)$. Let p_0 be a critical point of f. Then p_0 is a non-degenerate critical point if the second derivative $f''(p_0) \neq 0$, and degenerate if $f''(p_0) = 0$. Note that we often simply say that p_0 is either non-degenerate or degenerate.

Before we give the definitions of a degenerate and non-degenerate critical point it is necessary to define the symmetric [2] bilinear functional $H_f(p_0)$ on the tangent space of M at a critical point p_0 of f by setting each (i, j)-entry in $H_f(p_0)$ to be $\frac{\partial^2 f}{\partial x_i \partial x_j}$. In the expanded notation we have that

$$H_f(p_0) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2}(p_0) & & \cdots & & \frac{\partial^2 f}{\partial x_1 \partial x_m}(p_0) \\ & \ddots & & & \\ \vdots & & \frac{\partial^2 f}{\partial x_i \partial x_j}(p_0) & & \vdots \\ & & & \ddots & \\ \frac{\partial^2 f}{\partial x_m \partial x_1}(p_0) & & \cdots & & \frac{\partial^2 f}{\partial x_m^2}(p_0) \end{bmatrix}$$

This $m \times m$ matrix is called the *Hessian* of f at the critical point p_0. The number of negative eigenvalues is called the *index* of p_0. If the determinant of $H_f(p_0)$ is zero, then f is *degenerate.* If the determinant of $H_f(p_0)$ is non-zero, then f is *non-degenerate.* We note that degeneracy and non-degeneracy of f does not depend on the coordinate system. If $f : M \to \mathbb{R}$ is a smooth function whose critical points are all non-degenerate, then f is called a *Morse function.* Intuitively, a Morse function allows us to keep track of its non-degenerate critical points by mapping them to the real line. One can see how that may be useful. If not, keep reading.

Example 2.1. *Let $M = \mathbb{R}^3$ and let $f : M \to \mathbb{R}$ be a quadratic polynomial such that $f(x, y, z) = x^2+y^2+z^2$. It is easy to check that $f(x, y, z)$ is smooth about the coordinate system (x, y, z). Then $\mathbf{0} = (0, 0, 0)$ is the only critical point. The Hessian of f at $\mathbf{0}$ is the 3×3 matrix*

$$H_f(\mathbf{0}) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2}(\mathbf{0}) & \frac{\partial^2 f}{\partial x \partial y}(\mathbf{0}) & \frac{\partial^2 f}{\partial x \partial z}(\mathbf{0}) \\ \frac{\partial^2 f}{\partial y \partial x}(\mathbf{0}) & \frac{\partial^2 f}{\partial y^2}(\mathbf{0}) & \frac{\partial^2 f}{\partial y \partial z}(\mathbf{0}) \\ \frac{\partial^2 f}{\partial z \partial y}(\mathbf{0}) & \frac{\partial^2 f}{\partial z \partial x}(\mathbf{0}) & \frac{\partial^2 f}{\partial z^2}(\mathbf{0}) \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Once we have computed $H_f(\mathbf{0})$, it is easy to check that $\mathbf{0}$ is a non-degenerate point since $\det H_f(\mathbf{0}) = 6$ *is nonzero.*

Lemma 2.2 (Morse Lemma)**.** *Let M be an m-dimensional smooth manifold. If p_0 is a non-degenerate critical point of $f : M \to \mathbb{R}$ of index λ, then there exists a coordinate system $\mathbf{x} = (x_1, \ldots, x_m)$ about p_0 such that*

$$f(\mathbf{x}) = f(p_0) - x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_m^2.$$

We call the form above *standard form.* The proof for the Morse lemma will not be included here since the proof is relatively involved and does not give us much more insight or understanding. However, if one is interested, it can be found in [2]. The Morse lemma tells us that it is enough to look at the quadratic polynomials in order to understand a functions local behaviour near non-degenerate critical points.

2.1. **Handles & Handlebodies.** Now that we have an understanding of critical points, we begin looking at how we can understand the structure different parts of a manifold by looking at sets of points in the manifold which are "before" or "after" different critical values of a Morse function defined on the manifold.

[2] It is symmetric due to Clairaut's theorem.

If M is *closed* (i.e., compact without boundary) and $f : M \to \mathbb{R}$ is a Morse function, set

$$M_t = \{p \in M | f(p) \leq t\}$$
$$M_{[a,b]} = \{p \in M | a \leq f(p) \leq b\}.$$

Theorem 2.3. *Let M be a smooth manifold. Then M admits a Morse function $f : M \to \mathbb{R}$ such that M_a is compact for all $a \in \mathbb{R}$.*

So loosely what we get here is that if we look at a structure M where p_0 is the only non-degenerate critical point of $f : M \to \mathbb{R}$ and $a, b \in \mathbb{R}$ such that $a \leq f(p_0) \leq b$, then looking at M_a and M_b as level sets:

- M_a has the same topology for all $a \leq f(p_0)$,
- M_b has the same topology for all $b \geq f(p_0)$,
- and M_a and M_b have different topological structures.

The last item moves us into a fundamental theorem of Morse theory:

Theorem 2.4. *M_a is diffeomorphic to M_b ($M_a \cong M_b$) if there do not exist any critical values of a Morse function $f : M \to \mathbb{R}$ in the interval $[a, b] \subseteq \mathbb{R}$.*

Before introducing the notion of a handlebody, which is formed by attaching many handles to a disk we will define a handle and walk through the process of attaching a handle to an m-dimensional disk D^m.

Let M be a closed manifold and let $f : M \to \mathbb{R}$ be a Morse function with critical values organized in ascending order $c_0 < c_1 < \cdots < c_n$ corresponding to critical points $p_0, p_1, \ldots, p_n$ of f. We know there are only finitely many critical values by a well known corollary of the Morse lemma [1, Cor. 2.19]. We begin by explaining why one begins with an m-dimensional disk. If m-dimensional disk D^m let $\varepsilon > 0$ then we notice that D^m is diffeomorphic to the set

$$M_{c_0+\varepsilon} = \{\mathbf{x} | x_1^2 + \cdots + x_m^2 \leq \varepsilon\}. \tag{1}$$

We can then construct an analogous set for a critical value $c_i = f(p_i)$ where p_i is a critical point of f of index λ_i :

$$M_{c_i-\varepsilon} = \{\mathbf{x} | x_1^2 + \cdots + x_{\lambda_i} - x_{\lambda_i+1}^2 - \cdots - x_m^2 \geq \varepsilon\}.$$

Now consider the following set for $0 < \delta << \varepsilon$:

$$\{(x_1, \ldots, x_{\lambda_i}, x_{\lambda_i+1}, \ldots x_m) | x_1^2 + \cdots + x_{\lambda_i}^2 \leq \varepsilon \text{ and } x_{\lambda_i+1}^2 + \cdots + x_m^2 \leq \delta\}. \tag{2}$$

The set in (2) is diffeomorphic to $D^{\lambda_i} \times D^{m-\lambda_i}$, i.e., the direct product of the λ_i-dimensional disk and the $(m - \lambda_i)$-dimensional respectively. We call $D^{\lambda_i} \times D^{m-\lambda_i}$ a *λ_i-handle*.[3]

We can look at the "center" of each disk by taking the cross product of each disk with the zero vector:

$$D^{\lambda_i} \times \mathbf{0} = \{(x_1, \ldots, x_{\lambda_i}, 0, \ldots 0) | x_1^2 + \cdots + x_{\lambda_i}^2 \leq \varepsilon\}$$
$$\mathbf{0} \times D^{m-\lambda_i} = \{(0, \ldots, 0, x_{\lambda_i+1}, \ldots x_m) | x_{\lambda_i+1}^2 + \cdots + x_m^2 \leq \delta\}$$

called the *core* and the *co-core* respectively.[4]

We may attach the λ_i-handle to $M_{c_i-\varepsilon}$ along their respective boundaries by a smooth embedding

$$\varphi : \partial D^{\lambda_i} \times D^{m-\lambda_i} \to \partial M_{c_i-\varepsilon}. \tag{3}$$

called the *attaching map* where each point $p \in \partial D^{\lambda_i} \times D^{m-\lambda_i}$ is identified with the point $\varphi(p) \in \partial M_{c_i-\varepsilon}$.

The boundary of ∂D^{λ_i} is a $(\lambda_i - 1)$-dimensional sphere S^{λ_i-1}. So, $\partial D^{\lambda_i} \times D^{m-\lambda_i}$ is a $(\lambda_i - 1)$-sphere of $(m - \lambda_i)$-dimensional thickness. We call this the *attaching sphere*, thereby allowing us to rewrite 3 as

$$\varphi : S^{\lambda_i-1} \times D^{m-\lambda_i} \to \partial M_{c_i-\varepsilon}.$$

[3] Most texts drop the subscript i and assume the reader understands by context that the index λ corresponds to a specific critical point, and thereby call the object a "λ-handle" or even simply a "handle".

[4] We use the term co-core, because the co-core is the dual of the core, meaning they intersect transversely determining the thickness of the handle.

Now that we have attached the handle $D^{\lambda_i} \times D^{m-\lambda_i}$ to $M_{c_i-\varepsilon}$ along the boundaries, we have the object

$$M_{c_i-\varepsilon} \cup D^{\lambda_i} \times D^{m-\lambda_i}$$

which is diffeomorphic to $M_{c_i+\varepsilon}$ [1].

We construct a *handlebody* by beginning with a disk D^m, which is itself an m-dimensional handlebody. [5] Then, given an attaching map $\varphi_1 : \partial D^{\lambda_1} \times D^{m-\lambda_1} \to \partial D^m$, we attach a λ_1-handle to D^m so that

$$D^m \cup_{\varphi_1} D^{\lambda_1} \times D^{m-\lambda_1}$$

This gives us the handlebody, $\mathcal{H}(D^m; \partial_1)$. We continue this process by attaching λ_i-handles, $D^{\lambda_i} \times D^{d-\lambda_i}$ by attaching maps $\varphi_i : \partial D^{\lambda_i} \times D^{m-\lambda_i} \to \partial\mathcal{H}(D^m; \partial_1, \ldots, \partial_{i-1})$ to create the m-dimensional handlebody,

$$\mathcal{H}(D^m; \partial_1, \ldots, \partial_{i-1}, \partial_i) = D^m \cup_{\varphi_1} D^{\lambda_1} \times D^{m-\lambda_1} \cup_{\varphi_2} \cdots \cup_{\varphi_i} D^{\lambda_i} \times D^{m-\lambda_i}.$$

Note that all attaching maps are smooth embeddings of class C^∞ and the boundary of the core disk D^λ is the $(\lambda-1)$-dimensional sphere $S^{\lambda-1}$. So we maintain a smooth structure each time we attach a new handle.

3. Proof of Main Theorem

Main Theorem. [1, 2, 3, 4] *A compact manifold M has a handle decomposition given by a Morse function $f : M \to \mathbb{R}$ where there is a corresponding handle for each critical point of f. Moreover, the indices of the the handles and the indices of the critical points align.*

Proof. Let f have n distinct critical points $p_0, p_1, \ldots, p_n$ and let us order them in such a way that the critical values $c_0, c_1, \ldots, c_n$ are ordered in ascending order. For each p_i let λ_i denote the index of the critical point p_i. In order to prove our main theorem it is sufficient to prove for all $0 \leq i \leq n$ that $M_{c_i+\varepsilon}$ is a handlebody. Fix a gradient-like vector field on M for f.

We do so by induction on i: The base case, $i = 0$, was already shown that $M_{c_i+\varepsilon}$ as seen in (1) is diffeomorphic to the m-dimensional disk D^m, which is a handlebody by definition.

Assuming for our induction hypothesis that for $\varepsilon > 0$, $M_{c_{i-1}+\varepsilon} = \mathcal{H}(D^m; \varphi_1, \ldots, \varphi_{i-1})$ is a handlebody, we want to show that the set $M_{c_i+\varepsilon}$ is a handlebody $\mathcal{H}(D^m; \varphi_1, \ldots, \varphi_{i-1}, \varphi_i)$ as well.

We have ordered the critical values in ascending order, so that we know there does not exist any critical value in the interval $[c_{i-1} + \varepsilon, c_i - \varepsilon] \subseteq \mathbb{R}$. Therefore, by Theorem 2.4, $M_{c_{i-1}+\varepsilon} \cong M_{c_i-\varepsilon}$. Let us call this map ϕ. We find this diffeomorphism by letting $M_{c_{i-1}+\varepsilon}$ flow along the gradient vector field X until it aligns with $M_{c_i-\varepsilon}$. So then

$$\mathcal{H}(D^m; \varphi_1, \ldots, \varphi_{i-1}) = M_{c_i-1} \cong M_{c_{i-1}-\varepsilon}.$$

But we know that $M_{c_i+\varepsilon} \cong M_{c_i-\varepsilon} \cup_{\varphi_i} D^{\lambda_i} \times D^{m-\lambda_i}$ by attaching a λ_i-handle via the attaching map

$$\varphi_i : \partial D^{\lambda_i} \times D^{m-\lambda_i} \to \partial M_{c_i-\varepsilon}.$$

Therefore,

$$\mathcal{H}(D^m; \varphi_1, \ldots, \varphi_{i-1}) \cup_{\varphi_i} D^{\lambda_i} \times D^{m-\lambda_i} = \mathcal{H}(D^m; \varphi_1, \ldots, \varphi_{i-1}, \varphi_i).$$ [6]

Thus closes the induction. □

An extremely important corollary of the handlebody decomposition of a closed manifold was published in 1963 by Milnor.

Corollary 3.1 (Milnor [2])**.** *Let $M = \mathcal{H}(D^d; \varphi_1, \ldots, \varphi_k)$ be a d-dimensional handlebody with handles $D^{\lambda_i} \times D^{d-\lambda_i}$, and let*

$$\ell = \max\{j : D^{\lambda_j} \times D^{d-\lambda_j} \subseteq M\}.$$

Then M is homotopy equivalent to an ℓ-dimensional cell complex Γ.

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[5] If the reader is familiar with the concept of simplicial complexes, one can consider the construction of a handlebody roughly analogous to the construction of a simplicial complex. A simplicial complex is constructed by gluing many k-dimensional simplicies, whereas a handlebody is constructed by gluing together λ-handles.

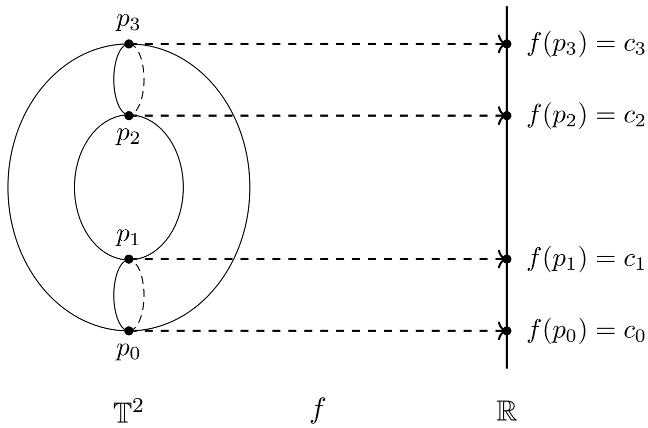
[6] We admit that one could be a be a bit more careful and say that the λ_i-handle is actually being attached to the boundary of the handlebody $\mathcal{H}(D^m; \varphi_1, \ldots, \varphi_{i-1})$ by the composit map $\phi^{-1} \circ \varphi_i$, where ϕ^{-1} is the inverse image of the diffeomorphism $\phi : M_{c_{i-1}+\varepsilon} \to M_{c_i-\varepsilon}$.

4. Example: The Torus ($\mathbb{T}^2$)

Let $M = \mathbb{T}^2$ (the torus of genus one) and let f be the height function on M. The torus has 4 critical points p_0, p_1, p_2, p_3 which give rise to four critical values $c_0, c_1, c_2, c_3 \in \mathbb{R}$. The following table contains the data we can collect on the critical points:

Critical Point	**Critical Value**	**Type**	**Index**
p_0	$f(p_0) = c_0$	minima	0
p_1	$f(p_1) = c_1$	saddlepoint	1
p_2	$f(p_2) = c_2$	saddlepoint	1
p_3	$f(p_3) = c_3$	maxima	2

We now demonstrate the four Morse function mappings from the critical points on the torus to the real line in the following diagram:



Let $t \in \mathbb{R}$, then if we slowly sweep $\mathbb{T}^2$ from V up we can understand the topology of M_t when t is contained in between the critical values in the following ways:

- If $t < c_0$, then M_t is vacuous.
- If $c_0 < t < c_1$, then M_t diffeomorphic to a disk (a 2-cell), with boundary diffeomorphic to a circle.
- If $c_1 < t < c_2$, then M_t is diffeomorphic to a cylinder, with boundary diffeomorphic to a circle.
- If $c_2 < t < c_3$, then M_t is diffeomorphic to a compact manifold of genus one with boundary component diffeomorphic to a circle.[7]
- If $c_3 < t$, then M_t is $\mathbb{T}^2$.

References

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Email address: `adams869@umn.edu`

[7] This is the torus minus the open, where the single boundary component is diffeomorphic to a circle.