A generalized coinvariant algebra

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1. Simplicial Complexes

2a. The coinvariant algebra A_n

2b. A combinatorial model for A_n

3a. A generalized coinvariant algebra A_n^Δ

3b. A combinatorial model for A_n^Δ

4. Applications

1 Simplicial Complexes

DEF A simplicial complex Δ on a set $V = \{1, \dots, n\}$ is a collection of subsets $F \subset V$ called faces which satisfy

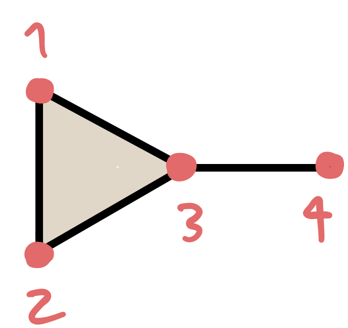
i) If $v \in V$, then $\{v\} \in \Delta$

ii) If $G \subseteq F$ and $F \in \Delta$, then $G \in \Delta$.

DEFs

- $\dim(F) = |F| - 1$
- A face $F \in \Delta$ s.t. $\nexists\, G \in \Delta$ so that $F \subseteq G$ is called a facet.
- If F is a maximal facet, then $\dim(\Delta) = \dim(F)$.

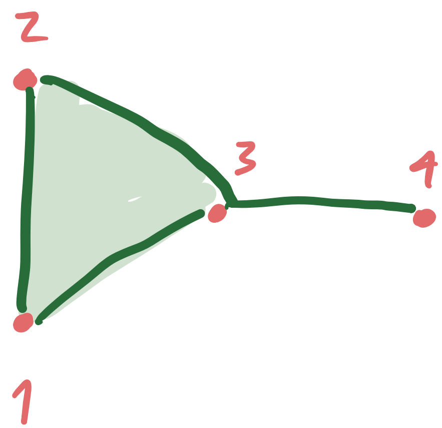
EX.

$\Delta =$ 

1

Ex.

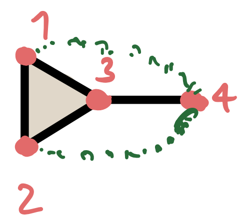
$$\underbrace{V = \{1, 2, 3, 4\}}^{\text{Vertex set}} \qquad \overbrace{\begin{array}{l} F_1 = \{1,2,3\} \\ F_2 = \{3,4\} \end{array}}^{\text{Facets}}$$



DEF The Stanley-Reisner ring of a simplicial complex Δ on a vertex set V is the quotient ring

$$\Bbbk[\Delta] = S/I_\Delta = \frac{\Bbbk[x_v \mid v \in V]}{\langle x_\sigma \mid \sigma \notin \Delta \rangle}$$

Ex 1 cont.

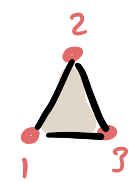


$$= \Delta$$

$$\frac{\Bbbk[x_1, x_2, x_3, x_4]}{\langle x_1 x_4, \ x_2 x_4 \rangle} = \Bbbk[\Delta]$$

Ex 2 (Simplex)

Let $\Delta = \Delta_2$ the 2-simplex.



$\Bbbk[x_1, x_2, x_3]$

(2a)

Combinatorics

$S = \Bbbk[x_1, \dots, x_n]$, $\deg(x_i) = 2$

DEF The elementary symmetric functions

$$e_1(x_1, \dots, x_n), \dots, e_n(x_1, \dots, x_n)$$

are given by

$$e_k(x_1, \dots, x_n) = \sum_{\substack{S \subset [n] \\ |S| = k}} x_S$$

DEF The coinvariant algebra:

$$A_n = S/I_n = \frac{\Bbbk[x_1, \dots, x_n]}{(e_1(x), \dots, e_n(x))}$$

Algebraic Geometry

L_i = a line bundle on $BU(n)$.

$x_i := c_1(L_i) \in H^2(BU(n), \mathbb{Z})$

is the first Chern class of L_i.

$$B_i : BU(1)^n \longrightarrow BU(n)$$

$$(B_i)^*(c_1(L_i)) = e_i(x).$$

$K := SU(n)$

$T :=$ maximal torus

$K/T :=$ flag variety

- The cohomology ring of K/T is A_n.

THM (Artin)

The substaircase monomials

$$\mathcal{B} = \{ x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \mid 0 \le \alpha_i \le i-1 \}$$

form a basis for A_n.

Substaircase monomials:

degree	monomial
0	1
1	x_2, x_3, x_4
2	$x_3^2, x_4^2,$ $x_2x_3, \ldots, x_3x_4$
3	$x_4^3, x_2x_3^2, x_2x_4^2,$ $x_3^2x_4, x_3x_4^2,$ $x_2x_3x_4$
$\vdots$	$\vdots$

EX

$$A_4 = \frac{\Bbbk[x_1, x_2, x_3, x_4]}{(e_1(x), e_2(x), e_3(x), e_4(x))}$$

$$e_1(x) = x_1 + x_2 + x_3 + x_4,$$

$$e_2(x) = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4,$$

$$e_3(x) = x_1x_2x_3 + x_1x_2x_4 + x_2x_3x_4,$$

$$e_4(x) = x_1x_2x_3x_4$$

Young Diagrams

DEF Let $j \in \mathbb{N}$. A sequence

$$\lambda_1 \geq \cdots \geq \lambda_t$$

with $\lambda_i \in \mathbb{N}$ such that

$$\sum_{i=1}^{t} \lambda_i = j$$

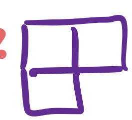
is called a <u>partition</u> and we denote it

$$\lambda = (\lambda_1, \ldots, \lambda_t) \vdash j.$$

$\xleftrightarrow{1:1}$

DEF A <u>young diagram</u> for an integer partition is a finite collection of boxes arranged in left-justified rows with row lengths in non-decreasing order.

EX. $\lambda = (2,1) \vdash 3$ because $2+1=3$.

EX. 2 1  $\#\text{boxes} = 3$

Young Diagrams cont.

integer partitions

$n = 3$

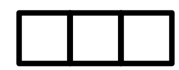
$3 + 0 = 3$

$2 + 1 = 3$

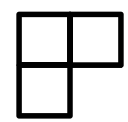
$1 + 1 + 1 = 3$

$\xleftrightarrow{1:1}$

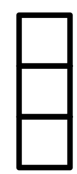
young diagram



(3)



(2, 1)



(1, 1, 1)

Monomials and Partitions

- Let $[n] := \{1, \dots, n\}$
- $\lambda = (\lambda_1, \dots, \lambda_t) \vdash j \leq n$
- $T \subseteq [n]$, $|T| = t$
- $\mathfrak{S}_T$ = symmetric group with permutation values in T
- $\sigma \in \mathfrak{S}_T$
- $\lambda^\sigma := (\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(t)})$ ← "reordering"
- A monomial $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \Bbbk[x_1, \dots, x_n]$ with $\alpha_{\sigma(i)} = \lambda_{\sigma(i)}$ or $\alpha_j = 0$ otherwise.

Ex. $\Bbbk[x_1, x_2, x_3, x_4, x_5, x_6]$

$$\lambda = (\lambda_1, \lambda_2, \lambda_3) = (5, 3, 1)$$

$$\sigma = (6\ 2\ 4) \in \mathfrak{S}_{\{2,4,6\}}$$

$$x_1^0 x_2^3 x_3^0 x_4^1 x_5^0 x_6^5 = x_2^3 x_4^1 x_6^5$$

A combinatorial method for computing the Artin basis

Setup

Step 1: Draw an $n \times n$ box

Step 2: Draw in a diagonal border:

SESE$\cdots$ES

Computation

For each $\lambda^{\sigma} = (\lambda_1, \ldots, \lambda_d) \vdash j$, $1 \leq j \leq \frac{1}{2}(n-1)n$,

that fit,

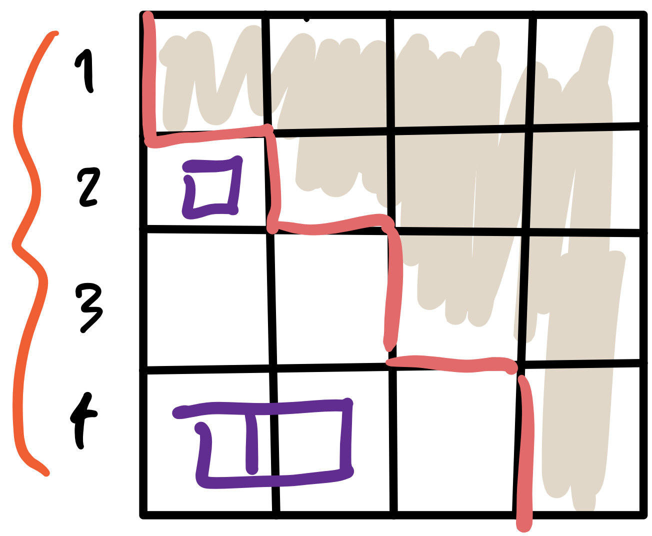
$$x_{\sigma(1)}^{\lambda_{\sigma(1)}} \cdots x_{\sigma(d)}^{\lambda_{\sigma(d)}}$$

is a substaircase monomial.

The set of all such monomials = $\left\{ \begin{array}{c} \text{substaircase} \\ \text{monomials} \end{array} \right\}$

Def A partition λ^{σ} fits if all $\lambda_{\sigma(i)}$ fit under the diagonal.

Ex. $n=4$



$\lambda^{\sigma} = (2,1) \rightsquigarrow x_2 x_4^2$

DEF Let S be a graded ring and let M be an $\mathbb{N}^d$-graded S-module then

$$M = \bigoplus_{b \in \mathbb{N}^d} M_b$$

has *Hilbert series*

$$\mathrm{Hilb}(M, q) := \sum_{b \in \mathbb{N}^d} \dim_{\mathbb{k}}(M_b) q^b$$

COR (Artin)

$$\mathrm{Hilb}(A_n, q) = [n]!_q$$

q-factorial

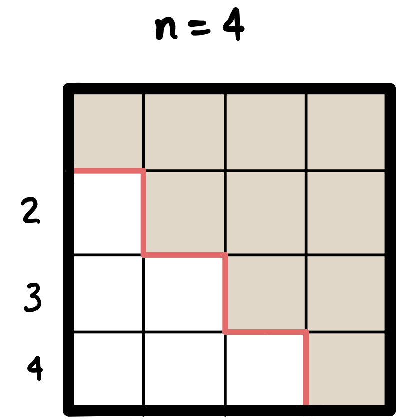
$$[n]!_q := [n]_q [n-1]_q \cdots [1]_q$$

q-integer

$$[n]_q := 1 + q + q^2 + \cdots + q^{n-1} = \frac{1-q^n}{1-q}$$

We can obtain the Hilbert series from the substaircase diagram?

j	λ	placements	total
0	ϕ		1
1	(1)	4 3 2	3
2	(2)	4 3	5
	(1,1)	4 3 3 2 2	
3	(3)	4	6
	(2,1)	4 3 3 2 4 2	
	(1,1,1)	4 3 2	
4	(3,1)	4 3 2	5
	(2,2)	4 3	
	(2,1,1)	4 3 3 4 2 2	
5	(3,2)	4 3	3
	(3,1,1)	4 3 2	
	(2,2,1)	4 3 2	
6	(3,2,1)	4 3 2	1



$$\mathrm{Hilb}(A_n, q) = 1 + 3q + 5q^2 + 6q^3 + 5q^4 + 3q^5 + 1q^6$$

THM (A.)

$$\mathrm{Hilb}(A_n, q) = \sum_{1 \le b \le n} \sum_{\lambda \vdash b} \left(\prod_{\substack{1 \le j \le \#\lambda \\ \text{s.t. } \lambda_j \ne \lambda_{j+1}}} \binom{n - \lambda_j - \#\{\lambda_i \mid \lambda_i > \lambda_j\}}{\#\{\lambda_i \mid \lambda_i = \lambda_j\}} q^{b} \right)$$

3a A generalized coinvariant algebra A_n^{Δ}

- Let $\Delta = \Delta_n$ be an n-simplex, then

$$\mathbb{k}[\Delta] = \mathbb{k}[x_1, \dots, x_n]$$

as an S_n-module, we obtain the coinvariant algebra

$$A_n = \frac{\mathbb{k}[\Delta]}{(e_1(x), \dots, e_n(x))}$$

with basis given by the substaircase monomials.

What if Δ is not simply a simplex?

- Let $\Delta = \Delta_n$ be an n-simplex, then

$$\Bbbk[\Delta] = \Bbbk[x_1, \dots, x_n]$$

as an S_n-module, we obtain the coinvariant algebra

$$\mathcal{A}_n = \frac{\Bbbk[\Delta]}{(e_1(x), \dots, e_n(x))}$$

with basis given by the substaircase monomials.

- Let Δ be a simplicial complex of dimension $d-1$ on n vertices with Stanley-Reisner ring $\Bbbk[\Delta] = S/I_\Delta$.

- The <u>universal system of parameters</u>

$$\theta = (\theta_1, \dots, \theta_d)$$

$$\theta_k = \sum_{\substack{F \in \Delta \\ \dim(F) = k-1}} x_F$$

is a homogeneous system of parameters.

- Define

$$\mathcal{A}_n^\Delta := \frac{\Bbbk[\Delta]}{(\theta_1, \dots, \theta_d)} = \frac{\Bbbk[x_1, \dots x_n]}{I_\Delta + I_n}$$

Question 1 Can we compute the Hilbert series for $\mathcal{A}_n^{\Delta}$?

Answer: Yes!

Question 2 Can we find an explicit basis for $\mathcal{A}_n^{\Delta}$?

Answer: Sometimes!

• When $I_\Delta = (m)$ a single monomial

$h_1[x_1,\dots,x_n],\dots,h_p[x_p,\dots,x_n],$

$$\left(\prod_{j=2}^{\ell+1}(x_{i_{k_j+1}}\cdots x_{i_{k_{j+1}}})\right) h_{i_{k_1}}[x_{i_{k_1}+1},\dots,x_n],\dots,\left(\prod_{j=2}^{\ell+1}(x_{i_{k_j+1}}\cdots x_{i_{k_{j+1}}})\right) h_{i_{k_1+1}-2}[x_{i_{k_1+1}-1},\dots,x_n],$$

$$\left(\prod_{j=2}^{\ell+1}(x_{i_{k_j+1}}\cdots x_{i_{k_{j+1}}})\right) h_{i_{k_2}}[x_{i_{k_2}+1},\dots,x_n],\dots,\left(\prod_{j=2}^{\ell+1}(x_{i_{k_j+1}}\cdots x_{i_{k_{j+1}}})\right) h_{i_{k_2+1}-2}[x_{i_{k_2+1}-1},\dots,x_n],\dots,$$

$$\left(\prod_{j=\ell+1}^{\ell+1}(x_{i_{k_j+1}}\cdots x_{i_{k_{j+1}}})\right) h_{i_{k_\ell}}[x_{i_{k_\ell}+1},\dots,x_n],\dots,\left(\prod_{j=\ell+1}^{\ell+1}(x_{i_{k_j+1}}\cdots x_{i_{k_{j+1}}})\right) h_{i_{k_\ell+1}-2}[x_{i_{k_\ell+1}-1},\dots,x_n]$$

$x_{i_t}h_{i_t-1}[x_{i_t},\dots,x_n],$

$h_{i_t}[x_{i_t+1},\dots,x_n],\dots,h_{n-1}[x_n].$

In certain circumstances, the following adjustments are made to the basis:

- *if $p = i_{k_1}$, then the basis also includes*

$$h_{i_{k_1}+1}[x_{i_{k_1}+1},\dots,\widehat{x_{i_t}},\dots,x_n],\dots,h_{i_t-1}[x_{i_t-1},\dots,\widehat{x_{i_t}},\dots,x_n].$$

- *if $i_{k_1} > 1$, then m itself is included in the basis as well.*
- *furthermore, when $i_{k_1} \geq t$, we substitute $h_t[x_t,\dots,x_n]$ with $h_t[x_t,\dots,x_n] - m$.*

Here,

- *$\{k_1,\dots,k_t\}$ is the jump set of m*
- *$p = i_{k_1}$ when $i_t < n$ and $p = i_t$ when $i_t = n$;*
- *and $\widehat{x_{i_t}}$ denotes the removal of x_{i_t} from the set of variables in which the reduced homogeneous function $h_m[x_m,\dots,x_n]$ is generated.*

the leading terms

$$x_1, x_2^2,\dots,x_p^p,$$

$$\left(\prod_{j=2}^{\ell+1}\mathbf{x}^{K_j}\right) x_{i_{k_1}+1}^{i_{k_1}},\dots,\left(\prod_{j=2}^{\ell+1}\mathbf{x}^{K_j}\right) x_{i_{k_1+1}-1}^{i_{k_1+1}-2},$$

$$\cdots$$

$$\mathbf{x}^{K_{\ell+1}} x_{i_{k_\ell}+1}^{i_{k_\ell}},\dots,\mathbf{x}^{K_{\ell+1}} x_{i_{k_\ell+1}-1}^{i_{k_\ell+1}-2}$$

$$x_{i_t}^{i_t}, x_{i_t+1}^{i_t},\dots,x_n^{n-1}.$$

- *If $p = i_{k_1}$, then we also have*

$$x_{i_{k_1}+1}^{i_{k_1}+1},\dots,x_{i_t-1}^{i_t-1}.$$

- *If $i_{k_1} > 1$, then m is also a leading term.*

Can we find a simpler way of computing

The graded reverse lexicographic order (grevlex)

Input: monomials m_1, m_2

Step 1. Compare degrees

i) If $\deg(m_1) > \deg(m_2)$ or $\deg(m_1) < \deg(m_2)$

then $m_1 >_{grevlex} m_2$ or $m_1 <_{grevlex} m_2$

ii) If $\deg(m_1) = \deg(m_2)$, then proceed to step 2.

Step 2. Use lexographic, $<_{lex}$ or $>_{lex}$:

- If $m_1 <_{lex} m_2$, then $m_1 >_{grevlex} m_2$.
- If $m_1 >_{lex} m_2$, then $m_1 <_{grevlex} m_2$.

EX. $x_3x_4x_1 > x_3x_4x_5 > x_3^2 > x_3x_5 > x_1 > x_3$

Base Case

- Let $\mathcal{A}_n^{\Delta} = \dfrac{\Bbbk[x_1,\dots,x_n]}{(e_1(\mathbf{x}),\dots,e_n(\mathbf{x})) + \underbrace{(m_1,\dots,m_\ell)}_{= I_\Delta}}$

- $\overbrace{m_1,\dots,m_\ell}^{\text{squarefree monomials}}$ can be of any degree!

- We ask for $m_i < m_{i+1}$ and that m_{i+1} is the least possible next choice that is still greater than m_i.

Ex. $n=5$: Let $I_{\Delta_1} = (\underline{x_4x_5}, \underline{x_3x_5}, x_1x_2x_5)$ ✓

$\xrightarrow{\text{least to greatest}}$

$I_{\Delta_2} = (x_4x_5, \cancel{x_2x_5}, x_2x_3x_4)$ ✗

THM (A.) Recipie for computing a minimal generating set for $I_n + I_\Delta$

1) Let $\ln(G_2) = \{ x_1^{a_1} \cdots x_n^{a_n} \mid 0 \le a_i \le i-1 \}$ ← substaircase monomials

2) For each m_j with $1 \notin \operatorname{supp}(m_j)$, $\ln(G_2) = \ln(G_2) \cup \{m_j\}$.

3) If $m_j = x_{i_1} \cdots x_{i_k} x_{i_{k+1}} \cdots x_{i_t}$, where k, is the least integer in $\operatorname{supp}(m_j)$ s.t. $i_k \neq i_{k+1} - 1$, then for all

The first place where the indices are not consecutive or "jump"

$i_{k+1} \le p \le i_t$,

- if $\tilde{m}_j^{(p)} = x_{i_{k+1}}^{\alpha_{i_{k+1}}} \cdots x_{i_t}^{\alpha_{i_t}} \in \ln(G_2)$

where

$$\alpha_{i_p} = i_p - 1 \text{ and } \alpha_{i_r} = 1, \quad i_{k+1} \le r \le i_t \ \& \ r \neq p,$$

then we reduce the degree of $\tilde{m}_j^{(p)}$ so that

$$\alpha_{i_p} = \alpha_{i_p} - 1.$$

- o.w. $\ln(G_2) = \ln(G_2) \cup \{ \tilde{m}_j^{(p)} \}$.

- Consider $\mathcal{A}_n^\Delta$ as an $\mathbb{N}$-graded ring

$$\Rightarrow \quad \mathcal{A}_n^\Delta = \bigoplus_{b \in \mathbb{N}} (\mathcal{A}_n^\Delta)_b , \qquad (\mathcal{A}_n^\Delta)_a (\mathcal{A}_n^\Delta)_b \subseteq (\mathcal{A}_n^\Delta)_{a+b}$$

- $(\mathcal{A}_n^\Delta)_b = \text{span}\left\langle \mathbf{x}^\alpha : \mathbf{y}^\beta \nmid \mathbf{x}^\alpha \text{ for all } \mathbf{y}^\beta \in \underline{\ln(C_2)} \right\rangle$

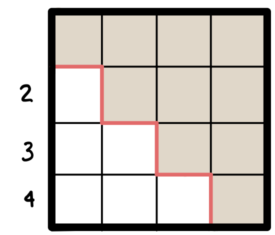
☆ **Punchline:** There is a combinatorial method for writing down these monomials explicitly.

A combinatorial model for computing a basis for $\mathcal{A}_n^\Delta$

Let $\mathcal{A}_n^\Delta = \dfrac{\mathbb{k}[x_1, \dots, x_n]}{I_n + I_\Delta}$

$$\mathcal{A}_4^\Delta = \frac{\mathbb{k}[x_1, x_2, x_3, x_4]}{I_n + (x_4 x_3)}$$

1) Draw an $n \times n$ box with diagonal boundary, á la $\mathcal{A}_n$:



2) Define a coloring

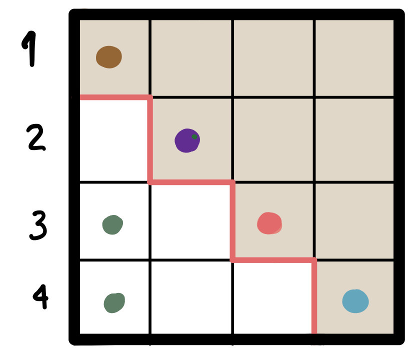
$$c : \mathrm{In}(G_2) \longrightarrow \{1, \dots, \#\mathrm{In}(G_n)\}.$$
$$m \longmapsto m^c$$

$$c : \overbrace{\{x_1^1, x_2^2, x_3^3, x_4^4, x_3x_4\}}^{\mathrm{In}(G_2)} \longrightarrow [5]$$

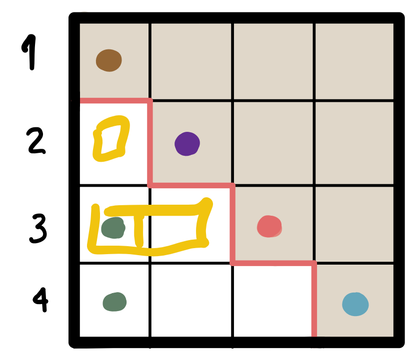
$$x_1^1 \longmapsto x_1^1$$
$$x_2^2 \longmapsto x_2^2$$
$$x_3^3 \longmapsto x_3^3$$
$$x_4^4 \longmapsto x_4^4$$
$$x_3x_4 \longmapsto x_3x_4$$

$\{x_1^1, x_2^2, x_3^3, x_4^4, x_3^1 x_4^1\}$

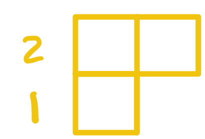
3) For each $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \ln(G_2)$, we place a ball of color m^c in row i, column α_i.



DEF Let $\lambda^{\sigma} = (\lambda_1, \ldots, \lambda_t)$, $\sigma \in S_T$, $T \subseteq \{2, \ldots, n\}$, $|T| = t$ be a partition that fits. If the young diagram corresponding to $\lambda_{\sigma(i)}$ which fits in row $\sigma(i)$ has a box which has a ball in it of color m^c, then we say that $\lambda_{\sigma(i)}$ has color m^c.



Ex. $\lambda = (2,1)$

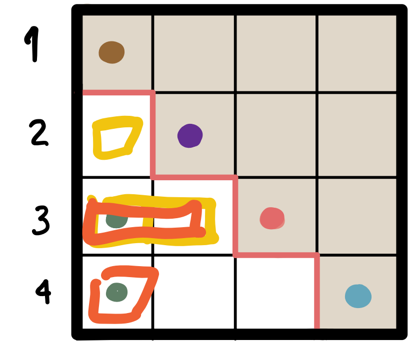


$\sigma \in S_{\{2,3,4\}}$, $\sigma = (3,2)$

4) To compute the basis $\mathcal{B}$ of Λ_n^{Δ} we require that any partition λ^{α}, that fits also have the property that

$$\#\{\text{parts of } \lambda \text{ with color } m^c\} = \#\operatorname{supp}(m)$$

for all $m \in \ln(C_2)$.



$\#\operatorname{supp}(x_3 x_3) = 2$

Ex. $\lambda = (2,1)$

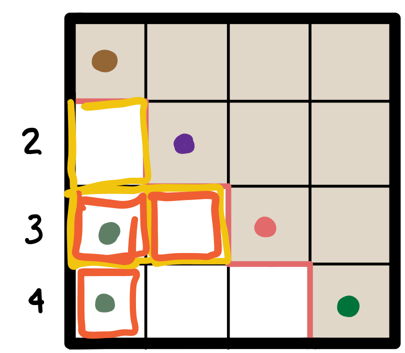
$\alpha_1 = (3,2)$ $\quad \lambda^{\alpha_1}$ fits

$\alpha_2 = (3,4)$ $\quad \lambda^{\alpha_2}$ does not fit.

5) For each λ^{α} that fits, λ^{α} defines a monomial:

$$x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

$\alpha_{\sigma(i)} = \lambda_{\sigma(i)}$ and $\alpha_j = 0$ o.w.



λ^{α_1}

$x_2^1 x_3^2 \in \mathcal{B}$

$x_3^2 x_4 \notin \mathcal{B}$

THM (A.) If $\lambda^\alpha = (\lambda_1, \dots, \lambda_t) \vdash b$ fits, then

$$x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

where $\alpha_{\sigma(i)} = \lambda_{\sigma(i)}$ and $\alpha_j = 0$ otherwise, is a generator of the b-th graded piece $(\mathcal{A}_n^\Delta)_b$ of $\mathcal{A}_n^\Delta$. All such monomials form a generating set for $(\mathcal{A}_n^\Delta)_b$. Moreover,

$$\mathcal{B} = \left\{ x_{\sigma(i_1)}^{\lambda_{\sigma(i_1)}} \cdots x_{\sigma(i_t)}^{\lambda_{\sigma(i_t)}} \;\middle|\; \lambda \vdash b,\ 1 \le b \le \tfrac{1}{2} n(n-1), \text{ and } \lambda^\alpha \text{ fits} \right\}$$

forms a basis for $\mathcal{A}_n^\Delta$.

This holds generally for all choices of Δ

choices of squarefree monomial

THM (A.)

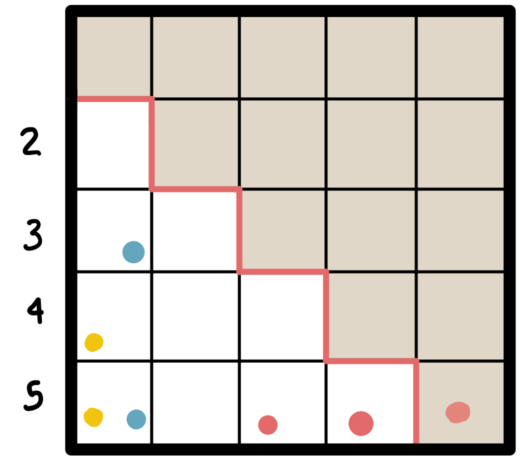
$$\mathcal{H}ilb(\mathcal{A}_n^{\Delta}, q) = \sum_{\substack{\lambda^{\sigma} \vdash 1, 2, \ldots, \frac{1}{2}(n-1)n \\ \text{s.t. } \lambda^{\sigma} \text{ fits}}} q^{\lambda^{\sigma}}$$

But $\mathcal{H}ilb(\mathcal{A}_n^{\Delta}, q)$ is invariant under $\mathfrak{S}_n \curvearrowright m$, $m \in I_{\Delta}$.

Thus, the above theorem holds for any choice of I_{Δ}!

Example

$n = 5, \quad I_\Delta = (x_5x_4,\ x_5x_3,\ x_5x_2x_1)$



$\mathrm{In}(G) = \{ x_5x_4,\ x_5x_3,\ x_5^3 \}$

degree	Hilbert series
1	$4q^{(1)}$
2	$3q^{(2)} + 4q^{(1,1)}$
	3 4 5 ; 2 / 3 4 5 ; 3 / 4
3	$1q^{(3)} + 5q^{(2,1)} + 1q^{(1,1,1)}$
	5 4 / 2 3 2 ; 3 / 4 2 ; 4 / 3 / 2
4	$0q^{(4)} + 2q^{(3,1)} + 1q^{(2,2)} + 2q^{(2,1,1)} + 0q^{(1,1,1,1)}$
	4 / 3 2 ; 4 / 3 ; 4 / 3 / 2 ; 3 / 4 / 2
5	$0q^{(5)} + 0q^{(4,1)} + 1q^{(3,2)} + 1q^{(3,1,1)} + 1q^{(2,2,1)} + 0q^{(2,1,1,1)} + 0q^{(1,1,1,1,1)}$
	4 / 3 ; 4 / 3 / 2 ; 4 / 3 / 2
6	$1q^{(3,2,1)}$
	4 / 3 / 2

$$\boxed{\mathrm{Hilb}(A_n^\Delta, q) = 1 + 4q + 7q^2 + 7q^3 + 5q^4 + 3q^5 + 1q^6}$$

Applications

- Let Δ be a simplicial complex of dimension $d-1$ & let $\mathbb{k}[\Delta] = S/I_\Delta$ be it's Stanley-Reisner ring.

- Let $\Theta = (\theta_1, \dots, \theta_d) \subseteq S$ be the universal system of parameters, i.e.

$$\theta_i = \sum_{\substack{F \in \Delta \\ \dim(F) = i-1}} x_F$$

$$\boxed{\begin{array}{l} \underline{\text{THM}}\,(A.)\ \text{Let } S \longrightarrow A = \mathbb{k}[z_1, \dots z_d], \\ \qquad\qquad \theta_i \longmapsto z_i \\ \text{then for } \mathbb{k}[\Delta] \text{ as an } A\text{-module} \\ \dim_{\mathbb{k}} \operatorname{Tor}_0^A(\mathbb{k}[\Delta], \mathbb{k})_b = \#\{\lambda^\sigma \vdash b \mid \lambda^\sigma \text{ fits}\} \end{array}}$$

Proof. $\operatorname{Tor}_0^A(\mathbb{k}[\Delta], \mathbb{k}) \cong S/I_\Delta \otimes_A \mathbb{k} \cong S/(I_n + I_\Delta) = \mathcal{A}_n^\Delta$.

$$\operatorname{Hilb}(\mathcal{A}_n^\Delta, 1)_b = \sum_{\substack{\lambda^\sigma \vdash b \\ \text{s.t. } \lambda^\sigma \text{ fits}}} 1^{\lambda^\sigma} = \#\{\lambda^\sigma \vdash b \mid \lambda^\sigma \text{ fits}\}. \quad \square$$

Thank you

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What does this mean geometrically?